LECTURE NOTE FOR ENGINEERING MATH.-II.

Second Semester-2020 (Left over portion.)

Integration of the following forms :.

NOTE Trigonometry Substitution :.

1. a^2-x^2, putting $x = a\sin\theta$ or $x = a\cos\theta$.
2. a^2+x^2, " $x = a\tan\theta$, or $x = a\cot\theta$.
3. x^2-a^2, " $x = a\sec\theta$ or $x = a\,\mathrm{cosec}\,\theta$.
4. $\frac{a-x}{a+x}$, " $x = a\cos 2\theta$.

Ex:- $\int \frac{1}{a^2+x^2}\,dx$

put → ~~a tan~~ $x = a\tan\theta$
$dx = a\sec^2\theta \cdot d\theta$.

$= \int \frac{1}{a^2+a^2\tan^2\theta} \cdot a\sec^2\theta \cdot d\theta$.

$= \int \frac{1}{a^2\sec^2\theta} \cdot a\sec^2\theta \cdot d\theta$

$= \frac{1}{a}\int d\theta$

$= \frac{1}{a} \cdot \theta + c$

$= \frac{1}{a} \cdot \tan^{-1} x/a + c$ Ⓐ.

Ex:- $\int \frac{1}{a^2-x^2}\,dx$

put → $x = a\sin\theta$
$dx = a\cos\theta\, d\theta$.

$= \int \frac{1}{a^2-a^2\sin^2\theta} \cdot a\cos\theta\, d\theta$.

$= \int \frac{1}{a^2\cos^2\theta} \cdot a\cos\theta\, d\theta$

$= \frac{1}{a} \cdot \int \sec\theta \cdot d\theta$.

$= \frac{1}{a}\, \log|\sec\theta + \tan\theta| + c$

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$$= \frac{1}{a}\log\left|\frac{a}{\sqrt{a^2-x^2}}+\frac{x}{\sqrt{a^2-x^2}}\right|+c.$$

$$= \frac{1}{a}\log\left|\frac{a+x}{\sqrt{(a+x)(a-x)}}\right|+c.$$

$$= \frac{1}{a}\cdot\log\left|\sqrt{\frac{a+x}{a-x}}\right|+c.$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \quad (A)$$

<u>2nd method</u>

$$\int\frac{1}{a^2-x^2}\,dx$$

$$= \int\frac{1}{(a+x)(a-x)}\,dx$$

$$= \int\frac{1}{2a}\left\{\frac{1}{a+x}+\frac{1}{a-x}\right\}dx$$

$$= \frac{1}{2a}\int\frac{1}{a+x}\,dx+\frac{1}{2a}\int\frac{1}{a-x}\,dx$$

$$= \frac{1}{2a}\log|a+x|-\frac{1}{2a}\log|a-x|+c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|+c \quad (A)$$

Ex:- $\int \frac{1}{x^2-a^2} dx$

$= \int \frac{1}{a^2\sec^2\theta - a^2} a\sec\theta . \tan\theta \, d\theta$, put → $x = a\sec\theta$
$dx = a\sec\theta . \tan\theta \, d\theta$.

$= \int \frac{1}{a\tan^2\theta} . a\sec\theta . \tan\theta \, d\theta$.

$= \frac{1}{a} \int \frac{\cos^2\theta}{\sin^2\theta} . \frac{1}{\cos\theta} . \frac{\sin\theta}{\cos\theta} . d\theta$

$= 1/a \int \operatorname{cosec}\theta . d\theta$

$= 1/a \log|\operatorname{cosec}\theta - \cot\theta| + c$.

$= 1/a \log\left|\frac{x}{\sqrt{x^2-a^2}} - \frac{a}{\sqrt{x^2-a^2}}\right| + c$

$= 1/a \log\left|\frac{x-a}{\sqrt{x^2-a^2}}\right| + c$

$= 1/a \log$

Definite Integral

Properties

1) $\int_a^b f(x)\,dx = \int_a^b f(y)\,dy = \int_a^b f(t)\,dt = \int_a^b f(u)\,du.$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

4. $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$

5. $\int_a^b \{f(x) \pm g(x)\}\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

6. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where $a<c<b$.

7. $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

8. $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(x) \text{ is even.} \\ 0, & \text{if } f(x) \text{ is odd.} \end{cases}$

9. $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$

Example :- ①

Prove that $\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx = \pi/4$.

Proof.

Let $I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx$

$$= \int_0^{\pi/2} \frac{\cos(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\, dx$$

$$= \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$$

Therefore $2I = \int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx + \int_0^{\pi/2} \frac{\sin x}{\sin x + \cos x}\, dx$

$$= \int_0^{\pi/2} \frac{\sin x + \cos x}{\sin x + \cos x}\, dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x \right]_0^{\pi/2}$$

$$= \pi/2 - 0 = \pi/2$$

$\therefore \quad I = \frac{\pi}{4}$.

Hence $\int_0^{\pi/2} \frac{\cos x}{\sin x + \cos x}\, dx = \frac{\pi}{4}$.

Example - 2

Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \pi/4.$$

<u>Proof</u>

Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)}+\sqrt{\cos(\pi/2 - x)}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

Now $2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} dx$$

$$= \left[x\right]_0^{\pi/2}$$

$$= \pi/2 - 0 = \pi/2.$$

or, $I = \pi/4$.

Hence $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \pi/4$

Example - 3 Prove that

$$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$$

<u>Proof</u>

Let $$I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cot(\pi/2 - x)}}{\sqrt{\cot(\pi/2 - x)} + \sqrt{\tan(\pi/2 - x)}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$$

Now $$2I = \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx + \int_0^{\pi/2} \frac{\sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cot x} + \sqrt{\tan x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx$$

$$= \int_0^{\pi/2} dx$$

$$= [x]_0^{\pi/2} = \pi/2 - 0$$

or, $I = \dfrac{\pi}{4}$.

Hence $$\int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x} + \sqrt{\tan x}}\, dx = \pi/4$$

Example [illegible]: Prove that

$$\int_0^{\pi/2} \log(\tan x)\, dx = 0$$

Proof Let $I = \int_0^{\pi/2} \log(\tan x)\, dx$

$$= \int_0^{\pi/2} \log\{\tan(\tfrac{\pi}{2} - x)\}\, dx$$

$$= \int_0^{\pi/2} \log(\cot x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{1}{\tan x}\right) dx$$

$$= \int_0^{\pi/2} \log(\tan x)^{-1}\, dx$$

$$= -\int_0^{\pi/2} \log(\tan x)\, dx$$

$$= -I$$

or, $2I = 0$

or, $I = 0$

Hence $\int_0^{\pi/2} \log(\tan x)\, dx = 0$.

Example-6 Prove that $\int_0^{\pi/2} \frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi^2}{16}$

Proof

$$I = \int_0^{\pi/2} \frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$= \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)\sin(\frac{\pi}{2}-x)\cos(\frac{\pi}{2}-x)}{\sin^4(\frac{\pi}{2}-x) + \cos^4(\frac{\pi}{2}-x)}\,dx$$

$$= \int_0^{\pi/2} \frac{(\frac{\pi}{2}-x)\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx - \int_0^{\pi/2} \frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$= \pi/2\int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx - I$$

or $$2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx$$

$$= \frac{\pi}{2}\cdot\int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x}{1 + \tan^4 x}\,dx$$

$$= \frac{\pi}{4}\cdot\left[\tan^{-1}(\tan^2 x)\right]_0^{\pi/2}$$

$$= \frac{\pi}{4}\left\{\frac{\pi}{2} - 0\right\} = \frac{\pi^2}{8}.$$

$\therefore I = \frac{\pi^2}{16}$. Hence $\int_0^{\pi/2} \frac{x\sin x \cdot \cos x}{\sin^4 x + \cos^4 x}\,dx = \frac{\pi^2}{16}$.

Evaluate: $\int_0^{\pi/2} \log(\tan x + \cot x)\, dx$

Solution :-

$$\int_0^{\pi/2} \log(\tan x + \cot x)\, dx$$

$$= \int_0^{\pi/2} \log\left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos x}\right) dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2}{\sin 2x}\right) dx$$

$$= \log 2 \left(\int_0^{\pi/2} dx\right) - \int_0^{\pi/2} \log(\sin 2x)\, dx$$

$$= \frac{\pi}{2} \log 2 - I_1 \quad \text{———— (1)}$$

Now $I_1 = \int_0^{\pi/2} \log(\sin 2x)\, dx$

Putting $2x = t$

$2\,dx = dt$

when $x = 0,\ t = 0$

$x = \pi/2,\ t = \pi$

$$= \int_0^{\pi} \log(\sin t)\, \frac{dt}{2}$$

$$= \frac{1}{2} \int_0^{\pi} \log(\sin t)\, dt$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log(\sin x)\, dx \qquad \because f(\pi - x) = f(x)$$

$$= \int_0^{\pi/2} \log(\sin x)\, dx$$

$$= -\frac{\pi}{2} \log 2 \quad \text{See Ex 9} \quad \text{———— (2)}$$

Putting the value of (2) in (1)

$$\int_0^{\pi/2} \log(\tan x + \cot x)\, dx = \pi \log 2.$$

Example:-

Prove that $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$

Proof Let $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}dx$

Putting $x = \tan\theta$ $\therefore dx = \sec^2\theta\, d\theta$

when $x=0,\ \theta=0$
$x=1,\ \theta=\pi/4$

$$= \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\tan(\pi/4-\theta)\right\} d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{\tan\pi/4-\tan\theta}{1+\tan\pi/4\cdot\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \ln\left\{1+\frac{1-\tan\theta}{1+\tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \ln 2\, d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \ln 2.\left[\theta\right]_0^{\pi/4} - I.$$

or, $2I = \pi/4 \ln 2$ $\therefore I = \frac{\pi}{8}\ln 2.$

Hence $\int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2.$

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$$\int_0^2 |x|\,dx$$

$$= \int_0^2 x\,dx \quad \{\because x>0 \ \therefore |x|=x$$

$$= \left[\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left(2^2-0^2\right) = \frac{4}{2} = 2.$$

Example $\int_{-3}^{0} |x|\,dx$

$$= \int_{-3}^{0} (-x)\,dx \quad \{\because x<0 \ \text{so} \ |x|=-x$$

$$= -\left[\frac{x^2}{2}\right]_{-3}^{0}$$

$$= -\frac{1}{2}(0-9)$$

$$= 9/2.$$

Example $\int_{-3}^{2} |x|\,dx$

$$= \int_{-3}^{0} |x|\,dx + \int_0^2 |x|\,dx$$

$$= \int_{-3}^{0} (-x)\,dx + \int_0^2 x\,dx \quad \left\{\because |x| = \begin{cases} x, & x>0 \\ -x & x<0 \end{cases}\right.$$

$$= \left[-\frac{x^2}{2}\right]_{-3}^{0} + \left[\frac{x^2}{2}\right]_0^2$$

$$= -\frac{1}{2}(0-9) + \frac{1}{2}(4-0)$$

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Differential Equation (1)

An equation that involves differentials or derivatives is known as a differential equation.

Order

The order of the highest ordered derivative occurring in it is known as the order of the differential equation.

Degree The degree of a differential equation is the highest power of highest order present in it if the equation in a form free from radicals or fraction in its derivative.

Example.

① $\dfrac{d^2y}{dx^2} + 3\left(\dfrac{dy}{dx}\right)^2 + 2y = 0$, order $= 2$; degree $= 1$

② $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{3/2} = \dfrac{d^2y}{dx^2}$

or, $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^3 = \left(\dfrac{d^2y}{dx^2}\right)^2$, order $= 2$, degree $= 2$

③ $\dfrac{dy}{dx} + \dfrac{5}{\frac{dy}{dx}} = 7x$

or, $\left(\dfrac{dy}{dx}\right)^2 + 5 = 7x\dfrac{dy}{dx}$, order $= 1$, degree $= 2$

④ $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$, order $= 1$, degree $= 1$

⑤ $\left\{\dfrac{1 + \left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)}\right\}^{3/2} = a$

or, $\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^3 = a^2\left(\dfrac{d^2y}{dx^2}\right)^3$ order $= 2$, degree $= 3$

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[illegible]

Form the differential equation by eliminating the arbitrary constant.

(i) $y^2 = Cx$

(ii) $y = C_1 e^{C_2 x}$

(iii) $ax^2 + by = 1$

(i) Given $y^2 = Cx$ ——— ①

Differentiating both sides w.r.to x, we get

$$\therefore \quad 2y \frac{dy}{dx} = C. \text{ ——— ②}$$

From ① and ② we get

$$y^2 = 2y \frac{dy}{dx} \cdot x$$

or, $\boxed{y = 2xy_1}$ where $y_1 = \frac{dy}{dx}$

which is the required differential equation.

(ii) Given $y = C_1 e^{C_2 x}$ ——— ①

Differentiating both sides w.r.to x, we get

$$y_1 = C_1 e^{C_2 x} \cdot C_2 = C_1 C_2 e^{C_2 x} \text{ ——— ②}$$

Again differentiating we get

$$y_2 = C_1 C_2 e^{C_2 x} \cdot C_2$$

$$= C_1 C_2^2 e^{C_2 x} \text{ ——— ③}$$

Now from ② we have

$$y_1^2 = C_1^2 C_2^2 e^{2C_2 x}$$

$$= (C_1 e^{C_2 x})(C_1 C_2^2 e^{C_2 x})$$

$$= y y_2$$

or, $\boxed{y_1^2 = y y_2}$ which is the required differential eq.

Example - 4

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}x} \cdot \tan^{-1}x}{1+x^2}$$

or, $dy = \frac{e^{\tan^{-1}x} \cdot \tan^{-1}x}{1+x^2} dx$

Integrating both sides we get

$$y = \int \frac{e^{\tan^{-1}x} \cdot \tan^{-1}x}{1+x^2} dx$$

Putting $\tan^{-1}x = t$

$\frac{1}{1+x^2} dx = dt$

$$= \int t \cdot e^t dt$$

$$= t\left(\int e^t dt\right) - \int \frac{d}{dt}(t)\left(\int e^t dt\right) dt.$$

$$= t \cdot e^t - \int 1 \cdot e^t dt$$

$$= te^t - e^t + c$$

or, $y = (\tan^{-1}x - 1) e^{\tan^{-1}x} + c$; which is the required general solution.

Example - 5. Solve $\frac{dy}{dx} = x/y$

$\boxed{\text{2nd Type} \quad \frac{dy}{dx} = f(x) \cdot g(y)}$

or, $y\,dy = x\,dx$

Integrating both sides we get

$$\int y\,dy = \int x\,dx$$

or, $\frac{y^2}{2} = \frac{x^2}{2} - \frac{c}{2}$

or, $\boxed{x^2 - y^2 = c}$ which is the required general solution.

Given $ax^2 + by = 1$ ——— (1)

Differentiating both sides w.r.t. x, we get

$$a \cdot 2x + b \cdot y_1 = 0 \text{ ——— } (2)$$

Again for differentiating (2) we get

$$2a + b \cdot y_2 = 0 \text{ ——— } (3)$$

Now from (2) and (3) we have

$$\frac{a}{b} = -\frac{y_1}{2x}$$

and $\frac{a}{b} = \frac{-y_2}{2}$

$\therefore \; -\frac{y_1}{2x} = \frac{-y_2}{2}$

or, $\boxed{y_1 = x y_2}$ which is the required diff. eq.

Example - (3). Solve $\frac{dy}{dx} = \log x$

1st type
$\frac{dy}{dx} = f(x)$

or, $dy = \log x . dx$

Integrating both sides we get

$$y = \int \log x . dx$$

$$= \log x \left(\int dx\right) - \int \frac{d}{dx}(\log x)\left(\int dx\right) dx$$

$$= x\log x - \int \frac{1}{x} . x \, dx$$

$$= x\log x - x + C$$

or, $\boxed{y = x(\log x - 1) + C}$ which is the required general solution.

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Example-6 Solve $\frac{dy}{dx} = e^{x-y}$

Given

$$\frac{dy}{dx} = e^{x-y} = \frac{e^x}{e^y}$$

or, $e^y dy = e^x . dx$

Integrating both sides we get

$$\int e^y dy = \int e^x dx$$

or, $\boxed{e^y = e^x + c}$ which is the required general solution.

Example-7. Solve $\ln\left(\frac{dy}{dx}\right) = 2x + 3y$.

Given $\ln\left(\frac{dy}{dx}\right) = 2x + 3y$

$\therefore \quad \frac{dy}{dx} = e^{2x+3y} = e^{2x} . e^{3y}$

or, $e^{-3y} dy = e^{2x} . dx$

Integrating both sides we get

$$\int e^{-3y} dy = \int e^{2x} dx$$

or, $-\frac{e^{-3y}}{3} = \frac{e^{2x}}{2} - \frac{c}{6}$

or, $-2e^{-3y} = 3e^{2x} - c$

or, $\boxed{3e^{2x} + 2e^{-3y} = c}$

which is the required general

Example - 8 Solve $x\frac{dy}{dx} + y = y^2$

Given $x\frac{dy}{dx} + y = y^2$

or, $x\frac{dy}{dx} = y(y-1)$

or, $\frac{dy}{y(y-1)} = \frac{dx}{x}$

Integrating both sides we get

$$\int \frac{1}{y(y-1)}dy = \int \frac{1}{x}dx$$

or, $\int\left(\frac{1}{(y-1)} - \frac{1}{y}\right)dy = \int \frac{1}{x}dx$

or, $\log(y-1) - \log y = \log x + \log c$

or, $\log\left(\frac{y-1}{y}\right) = \log cx$

or, $\therefore \frac{y-1}{y} = cx$

or, $y - 1 = cxy$

or, $\boxed{y(1-cx) = 1}$ which is the required general solution.

Example - 9. Solve

$\frac{d^2y}{dx^2} = k$, where k is a constant.

3rd Type

$\frac{d^2y}{dx^2} = f(x)$

Given $\frac{d^2y}{dx^2} = k$

or, $d\left(\frac{dy}{dx}\right) = k.dx$

Integrating both sides we get

$$\frac{dy}{dx} = \int k\,dx$$

$= kx + c_1$, where c_1 is the constant of integration.

or, $dy = (Kx + c_1)\,dx$

Again integrating both sides we get

$$y = \int (Kx + c_1)\,dx$$

$$= K.\frac{x^2}{2} + c_2 x + c_2$$

or, $\boxed{y = \frac{K}{2}x^2 + c_2 x + c_1}$ which is the required general solution.

Example-10. Solve $\frac{d^2y}{dx^2} = \sec^2 x$.

Given $\frac{d^2y}{dx^2} = \sec^2 x$

or, $d\left(\frac{dy}{dx}\right) = \sec^2 x \,.\, dx$

Integrating both sides we get

$$\frac{dy}{dx} = \int \sec^2 x\, dx$$

$$= \tan x + C_1$$

or, $dy = (\tan x + c_1)\,dx$

Again integrating both sides we get

$$y = \int (\tan x + c_1)\,dx$$

$$= \log(\sec x) + C_1 x + c_2$$

or, $\boxed{y = \log(\sec x) + c_1 x + c_2}$, which is the required general solution

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Linear Equation

A differential equation of the form

$$\frac{dy}{dx} + P(x)\cdot y = Q(x)$$

is called a linear differential equation of first order.

$$\boxed{I.F = e^{\int P(x)dx}}$$

Example - 1

Solve $x\frac{dy}{dx} + 2y = 4x^2$.

Solution

Given $x\frac{dy}{dx} + 2y = 4x^2$

or, $\frac{dy}{dx} + \frac{2}{x}\cdot y = 4x$ ——— (1)

The above differential equation is in the form.

$$\frac{dy}{dx} + P(x)\cdot y = Q(x). \text{ ——— (2)}$$

which is a linear diff. eq of first order.

Here $P(x) = \frac{2}{x}$

$$I.F = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2.$$

Multiplying with the I.F to the eq (1) we get

$$x^2\left(\frac{dy}{dx} + \frac{2}{x}\cdot y\right) = 4x^3.$$

$$\boxed{\frac{d}{dx}(y\cdot x^2) = 4x^3}$$

$$d(x^2\cdot y) = 4x^3\cdot dx$$

Integrating we get

$$x^3 y = \int 4x^3\, dx$$
$$= x^4 + c$$

$y = x + cx^{-3}$, which is the required general solution.

Example - 2 Solve

$$(1+y^2)\, dx = (\tan^{-1} y - x)\, dy.$$

Solution. Given $(1+y^2)\, dx = (\tan^{-1} y - x)\, dy$

or, $$\frac{dx}{dy} = \frac{\tan^{-1} y}{1+y^2} - \frac{x}{1+y^2}$$

or, $$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2} \quad \text{--- (1)}$$

The above diff. eq is in the form

$$\boxed{\frac{dx}{dy} + P(y) \cdot x = Q(y)} \quad \text{--- (2)}$$

which is a linear diff. eq of first order.

Here $P(y) = \dfrac{1}{1+y^2}$

$$\therefore \; I.F = e^{\int P(y)\, dy}$$
$$= e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y} \quad \text{--- (3)}$$

Multiplying the I.F with eq (1) we get

$$e^{\tan^{-1} y} \left(\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x \right) = \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y}$$

or, $\frac{d}{dy}\left(e^{\tan^{-1}y}\cdot x\right) = \frac{\tan^{-1}y \cdot e^{\tan^{-1}y}}{1+y^2}$

or, $d\left(e^{\tan^{-1}y}\cdot x\right) = \frac{\tan^{-1}y \cdot e^{\tan^{-1}y}}{1+y^2}\cdot dy$

Integrating both sides we get

$$x\cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1}y \cdot e^{\tan^{-1}y}}{1+y^2}\cdot dy$$

Putting $\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$= \int t\cdot e^t\,dt$$

$$= t\cdot e^t - \int 1\cdot e^t\,dt$$

$$= t\cdot e^t - e^t + c$$

$$= (\tan^{-1}y - 1)\,e^{\tan^{-1}y} + c$$

or, $\boxed{x = \tan^{-1}y - 1 + c\,e^{-\tan^{-1}y}}$

which is the required general solution.

Example-3. Solve.

$$x\log x\cdot\frac{dy}{dx} + y = 2\log x.$$

Solution:- Given

$$x\log x\,\frac{dy}{dx} + y = 2\log x$$

or, $\frac{dy}{dx} + \frac{1}{x\log x}\cdot y = \frac{2}{x}$

The above diff. eq is in the form

$\frac{dy}{dx} + P(x)\cdot y = Q(x)$, which is a Linear diff. eq of first order.

Here $P(x) = \frac{1}{x\log x}$

$\therefore$ I.F. $= e^{\int P\,dx}$

$= e^{\int \frac{1}{x\log x}dx}$

$= e^{\log(\log x)} = \log x$

Multiplying both side with eq ① we have

$$\log x\left[\frac{dy}{dx} + \frac{1}{x\log x}\cdot y\right] = \frac{2}{x}\cdot\log x$$

or, $$\frac{d}{dx}(\log x\cdot y) = \frac{2\log x}{x}$$

$\Rightarrow$ $$d(y\log x) = \frac{2\log x}{x}\,dx$$

Integrating both sides we get

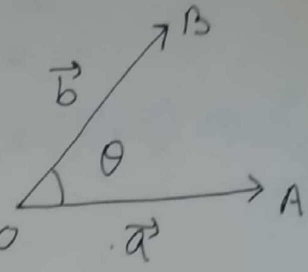
$$y\log x = \int\frac{2\log x}{x}\,dx$$

$$= 2\cdot\frac{(\log x)^2}{2} + C$$

$\therefore$ $$\boxed{y\log x = (\log x)^2 + C}$$

which is the required general Solution.

Dot Product or Scalar Product



$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$
$$= ab\cos\theta,$$

where θ is the angle between the two vectors $\vec{a}$ and $\vec{b}$ originating from a common point.

Note

1. If $\vec{a}$ and $\vec{b}$ are perpendicular then $\vec{a}\cdot\vec{b} = 0$ $\{\because \theta = 90^\circ$

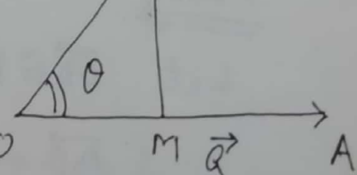
2. If $\vec{a}\cdot\vec{b} = 0$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ is $\perp$r to $\vec{b}$.

3. If $\vec{a}$ and $\vec{b}$ are like vectors then
$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos 0 \quad \{\because \theta = 0.$$
$$= ab$$
If $\vec{a}$ and $\vec{b}$ are unlike vectors then
$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos 180^\circ \quad \{\because \theta = 180$$
$$= -ab$$

4. $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
$$\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$$

5. $\vec{a}^{\,2} = \vec{a}\cdot\vec{a} = |\vec{a}|^2 = a^2.$

Scalar Projection of $\vec{b}$ on $\vec{a}$



$$= OM$$
$$= |\vec{OM}|$$
$$= |\vec{b}|\cos\theta$$
$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|} \quad \{\because \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta.$$

$$\cos\theta = \frac{OM}{OB} = \frac{|\vec{OM}|}{|\vec{OB}|}$$
$$\therefore |\vec{OM}| = |\vec{b}|\cos\theta.$$

Vector Projection of $\vec{b}$ on $\vec{a}$

$= \vec{OM}$

$= |\vec{OM}|\ \hat{OM}$ $\quad \left[\because \hat{a} = \dfrac{\vec{a}}{|\vec{a}|}\right.$

$= \left(\dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right)\left(\dfrac{\vec{a}}{|\vec{a}|}\right)$ $\quad \left[\because |\vec{OM}| = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right.$ and $\hat{OM} = \hat{a}$.

$= \dfrac{(\vec{a}\cdot\vec{b})\,\vec{a}}{|\vec{a}|^2}$

Also we write

$\boxed{\vec{a}\cdot\vec{b} = |\vec{a}|\ (\text{Scaler Projection of } \vec{b} \text{ on } \vec{a})}$

Properties.

1. Commutative law :– $\quad \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$
2. Dot product is not associative but it is associative with respect to a scalar

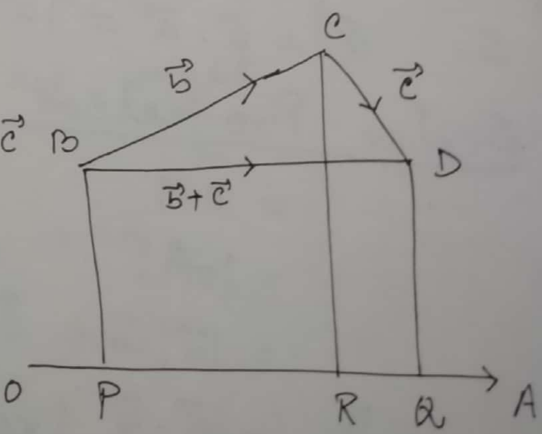
$$\lambda(\vec{a}\cdot\vec{b}) = (\lambda\vec{a})\cdot\vec{b} = \vec{a}\cdot(\lambda\vec{b})$$

3. Distributive law

$$\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$$

3. Distributive law. $\boxed{\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}}$

Proof.



Let $\vec{OA} = \vec{a}$, $\vec{BC} = \vec{b}$ and $\vec{CD} = \vec{c}$

$\therefore \vec{BD} = \vec{BC} + \vec{CD}$

$= \vec{b} + \vec{c}$

Draw the perpendiculars from the points B, D and C on the vector $\vec{a}$, which meet at the pts P, Q and R respect. ...

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L.H.S = $\vec{a} \cdot (\vec{b}+\vec{c})$

$= \vec{OA} \cdot \vec{BD}$

$= |\vec{a}|$ (scalar projection of $\vec{BD} = (\vec{b}+\vec{c})$ on $\vec{a}$)

$= |\vec{a}| (PQ)$

$= |\vec{a}| (PR + RQ)$

$= |\vec{a}| (PR) + |\vec{a}| (RQ)$

$= |\vec{a}|$ (scalar projection of $\vec{b}$ on $\vec{a}$)
$+ |\vec{a}|$ (scalar projection of $\vec{c}$ on $\vec{a}$)

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

= R.H.S.

Second method

Let $\vec{a}, \vec{b}, \vec{c} \in V_3$, where V_3 be the set of all three dimensional vectors.

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$
$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

L.H.S = $\vec{a} \cdot (\vec{b}+\vec{c})$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \{(b_1+c_1)\hat{i} + (b_2+c_2)\hat{j} + (b_3+c_3)\hat{k}\}$

$= a_1(b_1+c_1) + a_2(b_2+c_2) + a_3(b_3+c_3)$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

$= \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

= R.H.S.

NOTE ① Let $\vec{a} = a_1\hat{i} + b_1\hat{j} + b_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\therefore \vec{a}\cdot\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k})\cdot(b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$

$$= a_1 b_1\, \hat{i}\cdot\hat{i} + a_1 b_2\, \hat{i}\cdot\hat{j} + a_1 b_3\, \hat{i}\cdot\hat{k} + a_2 b_1\, \hat{j}\cdot\hat{i}$$

$$+ a_2 b_2\, \hat{j}\cdot\hat{j} + a_2 b_3\, \hat{j}\cdot\hat{k} + a_3 b_1\, \hat{k}\cdot\hat{i} + a_3 b_2\, \hat{k}\cdot\hat{j}$$

$$+ a_3 b_3\, \hat{k}\cdot\hat{k}$$

$$\boxed{\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3}$$

② $\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$

or, $$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$$

$$= \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Hence $$\theta = \cos^{-1}\left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\,\sqrt{b_1^2 + b_2^2 + b_3^2}}\right)$$

3. For the vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

(i) $(\vec{a}\cdot\vec{b})$ is a scaler quantity

(ii) $(\vec{a}\cdot\vec{b})\vec{c}$ is a vector quantity.

(iii) $(\vec{a}\cdot\vec{b})(\vec{c}\cdot\vec{d})$ is a scaler quantity.

(iv) $(\vec{a}\ \vec{b})$ is undefined.

(v) $5\cdot\vec{a}$ is undefined.

(vi) $5\times\vec{a}$ is undefined.

Ex-2 Page-227 Find the value of x, so that the vectors $\vec{a} = x\hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{b} = -x\hat{i} + x\hat{j} + 2\hat{k}$ are perpendicular.

Solution:- Given $\vec{a} = x\hat{i} - 3\hat{j} + 5\hat{k}$

and $\vec{b} = -x\hat{i} + x\hat{j} + 2\hat{k}$

Now $\vec{a}\cdot\vec{b} = (x\hat{i} - 3\hat{j} + 5\hat{k})\cdot(-x\hat{i} + x\hat{j} + 2\hat{k})$

$= -x^2 - 3x + 10$.

Since it is given $\vec{a}$ is $\perp r$ to $\vec{b}$, therefore

$\vec{a}\cdot\vec{b} = 0$

So $-x^2 - 3x + 10 = 0$

or, $(x+5)(x-2) = 0$

$\therefore$ $x = 2$ or $x = -5$.

No.5. Find the scalar and vector projection of $\vec{a}$ on $\vec{b}$; where $\vec{a} = \hat{i} - \hat{j} - \hat{k}$, $\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$

Solution:- Given $\vec{a} = \hat{i} - \hat{j} - \hat{k}$

$\vec{b} = 3\hat{i} + \hat{j} + 3\hat{k}$

Scalar projection of $\vec{a}$ on $\vec{b}$

$$= \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$$

$$= \frac{(\hat{i} - \hat{j} - \hat{k})\cdot(3\hat{i} + \hat{j} + 3\hat{k})}{\sqrt{3^2 + 1^2 + 3^2}} = \frac{3-1-3}{\sqrt{9+1+9}} = \frac{-1}{\sqrt{19}}$$

Vector projection of $\vec{a}$ on $\vec{b}$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|}\right)\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= \left(\frac{-1}{\sqrt{19}}\right)\left(\frac{3\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{19}}\right)$$

$$= \frac{-(3\hat{i} + \hat{j} + 3\hat{k})}{19}$$ Ans

NO. 7 If $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$, show that $|\vec{a}|=|\vec{b}|$

Solution:- Given

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$$

or, $\vec{a}\cdot(\vec{a}-\vec{b})+\vec{b}\cdot(\vec{a}-\vec{b})=0$

or, $\vec{a}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=0$

or, $\vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{b}=0$ $\{\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

or, $|\vec{a}|^2=|\vec{b}|^2$

$\therefore$ $|\vec{a}|=|\vec{b}|$

NO. 8(c). If $\vec{a}$ and $\vec{b}$ are perpendicular vectors, then show that $(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2$.

Solution:- L.H.S $=(\vec{a}+\vec{b})^2$

$=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$=(\vec{a}+\vec{b})\cdot\vec{a}+(\vec{a}+\vec{b})\cdot\vec{b}$

$=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}$

$=|\vec{a}|^2+|\vec{b}|^2$ $\{\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}=0$ as $\vec{a}$ is $\perp$r to $\vec{b}$.

R.H.S $=(\vec{a}-\vec{b})^2$

$=(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$

$=(\vec{a}-\vec{b})\cdot\vec{a}-(\vec{a}-\vec{b})\cdot\vec{b}$

$=\vec{a}\cdot\vec{a}-\vec{b}\cdot\vec{a}-\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}$

$=|\vec{a}|^2+|\vec{b}|^2$ $\{\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}=0$

Hence $(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2$.

ii) Prove that two vectors are perpendicular iff $|\vec{a}+\vec{b}|^2=|\vec{a}|^2+|\vec{b}|^2$.

Solution:- $|\vec{a}+\vec{b}|^2=(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})$

$=(\vec{a}+\vec{b})\cdot\vec{a}+(\vec{a}+\vec{b})\cdot\vec{b}$

$=\vec{a}\cdot\vec{a}+\vec{b}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}$

$=|\vec{a}|^2+|\vec{b}|^2+2\vec{a}\cdot\vec{b}$ —— (1) $\{\because \vec{a}\cdot\vec{b}=\vec{b}\cdot\vec{a}$

Case-I Given $\vec{a}$ is $\perp r$ to $\vec{b}$

$\therefore \quad \vec{a}\cdot\vec{b} = 0$ ——— (2)

Putting the value of (2) in (1)

$$|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2.$$

Case-II Given $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2$ ——— (3)

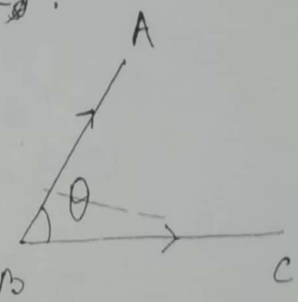
Putting the value of (3) in (1), we get

$$2\vec{a}\cdot\vec{b} = 0$$

or, $\vec{a}\cdot\vec{b} = 0$

Hence $\vec{a}$ is $\perp r$ to $\vec{b}$.

NO. 3 If A, B, C are points (1,0,2) (0,3,1) and (5,2,0) respectively, find $m\angle ABC$.



Solution: Given $\vec{OA} = \hat{i} + 2\hat{k}$

$\vec{OB} = 3\hat{j} + \hat{k}$

$\vec{OC} = 5\hat{i} + 2\hat{j}$

Now $\vec{BA} = \vec{OA} - \vec{OB} = \hat{i} - 3\hat{j} + \hat{k}$

$\vec{BC} = \vec{OC} - \vec{OB} = 5\hat{i} - \hat{j} - \hat{k}$

$$\cos(m\angle ABC) = \frac{\vec{BA}\cdot\vec{BC}}{|\vec{BA}|\,|\vec{BC}|}$$

$$= \frac{(\hat{i} - 3\hat{j} + \hat{k})\cdot(5\hat{i} - \hat{j} - \hat{k})}{\sqrt{1+9+1}\,\sqrt{25+1+1}}$$

$$= \frac{5+3-1}{\sqrt{11}\,\sqrt{27}} = \frac{7}{3\sqrt{33}}$$

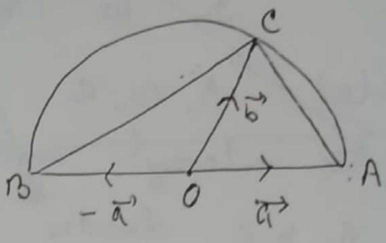
$$\therefore \quad m\angle ABC = \cos^{-1}\left(\frac{7}{3\sqrt{33}}\right)$$

Example:-

Prove by vector method that an angle incribed in a semi-circle is a right angle.

Proof.

Let us consider a semi-circle OACB whose Centre is at the origin, i.e, 'O'.



Again let $\vec{OA} = \vec{a}$ and $\vec{OC} = \vec{b}$

$\therefore \quad \vec{OB} = -\vec{a}$

Also $\vec{CA} = \vec{OA} - \vec{OC} = \vec{a} - \vec{b}$

$\vec{CB} = \vec{OB} - \vec{OC} = -\vec{a} - \vec{b}$

Now $\vec{CA} \cdot \vec{CB} = (\vec{a} - \vec{b}) \cdot (-\vec{a} - \vec{b})$

$= (\vec{a} - \vec{b}) \cdot (-\vec{a}) + (\vec{a} - \vec{b}) \cdot (-\vec{b})$

$= -\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b}$

$= |\vec{b}|^2 - |\vec{a}|^2 \qquad \{\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

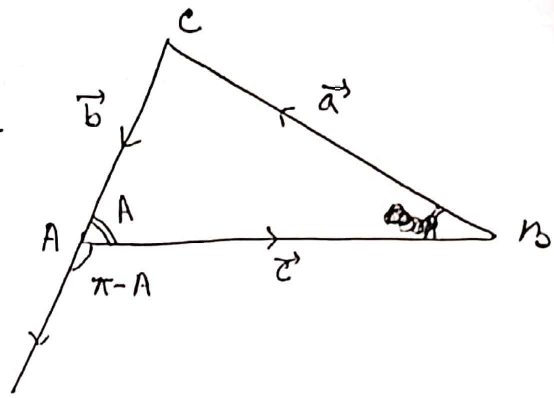
$= 0 \qquad \{\because |\vec{a}| = |\vec{b}| = \text{radius}.$

Hence the angle incribed in a semi-circle is a right angle.

Example

Derive the cosine law for plane triangle by vector methods.

Proof



In the triangle ABC, let A, B, C denote the interior angles and a, b, c denotes the ~~opp~~ sides opposite to them.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ denote the vectors $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ respectively. Then

$$\vec{a}+\vec{b}+\vec{c}=\vec{0}$$

or, $$\vec{a} = -(\vec{b}+\vec{c})$$

Squaring both sides we get

$$\vec{a}\cdot\vec{a} = (\vec{b}+\vec{c})\cdot(\vec{b}+\vec{c})$$

$$= \vec{b}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{b} + \vec{c}\cdot\vec{c}$$

$$= |\vec{b}|^2 + 2\vec{b}\cdot\vec{c} + |\vec{c}|^2 \qquad \{\because \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{b}$$

or, $$a^2 = b^2 + c^2 + 2bc\,\cos(\pi - A)$$

$$\{\because |\vec{a}| = a$$

or, $$a^2 = b^2 + c^2 - 2bc\cos A.$$

$$\therefore \cos A = \frac{b^2+c^2-a^2}{2bc}$$, which is the cosine law for plane triangle.

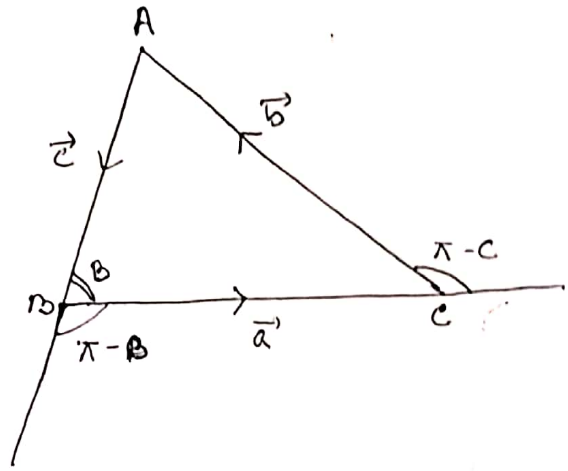
Similarly we can deduce the other two relations.

$$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

and $$\cos C = \frac{a^2+b^2-c^2}{2ab}.$$

Example :- ~~Derive~~ the projection formulae for plane triangle by vector method.

Proof. In the triangle ABC,



let A, B, C denote the exterior angles and a, b, c denote the side opposite to them.

Let $\vec{a}$, $\vec{b}$, $\vec{c}$ denote the vectors $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ respectively.

$$\therefore \quad \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

or, $\vec{a} = -(\vec{b} + \vec{c})$

Now $\vec{a}\cdot\vec{a} = -\vec{a}\cdot(\vec{b} + \vec{c})$

$= -\vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{c}$

or, $|\vec{a}|^2 = -|\vec{a}||\vec{b}|\cos(\pi - C) - |\vec{a}||\vec{c}|\cos(\pi - B)$

or, $a^2 = ab\cos C + ac\cos B$

or $a = b\cos C + c\cos B$

which is the projection formula for plane triangle.

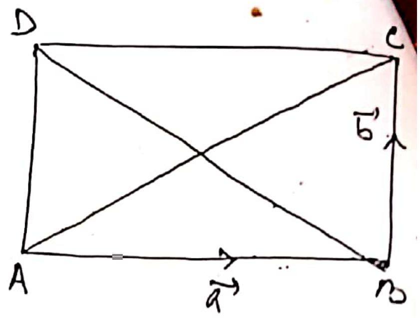
Similarly we can derive other two relation.

$$b = c\cos A + a\cos C$$

and $c = a\cos B + b\cos A$.

Example

The parallelogram whose diagonals are equal is a rectangle.



Proof

Let us consider a parallelogram ABCD whose sides $\vec{AB} = \vec{a}$ and $\vec{BC} = \vec{b}$.

$$\therefore \vec{AD} = \vec{BC} = \vec{b}.$$

Now $\vec{AC} = \vec{AB} + \vec{BC} = \vec{a} + \vec{b}$

and $\vec{DB} = \vec{AB} - \vec{AD} = \vec{a} - \vec{b}$

Given the diagonals are equal

i.e, $|\vec{AC}| = |\vec{DB}|$

or, $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}|$

Squaring both sides we get

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b}) = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$$

or, $\vec{a}\cdot\vec{a} + \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a} - \vec{a}\cdot\vec{b} - \vec{b}\cdot\vec{a} + \vec{b}\cdot\vec{b}$

or, $2\vec{a}\cdot\vec{b} = -2\vec{a}\cdot\vec{b}$ $\left\{ \because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} \right.$

or, $4\vec{a}\cdot\vec{b} = 0$

or, $\vec{a}\cdot\vec{b} = 0$

$\therefore$ $\vec{a}$ is perpendicular to $\vec{b}$.

Hence $\angle ABC = 90°$

Therefore the parallelogram whose diagonals are equal is a rectangle.

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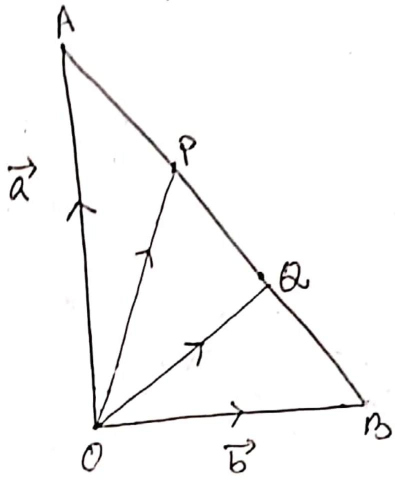
Example:

In a triangle AOB, $m\angle AOB = 90^\circ$. If P and Q are the points of trisection of AB, prove that $OP^2 + OQ^2 = \frac{5}{9} AB^2$.



Let us consider a triangle AOB whose $m\angle AOB = 90^\circ$ and P and Q are the pts of trisection of AB.

Let $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$

Here $\frac{AP}{PB} = \frac{1}{2}$ and $\frac{AQ}{QB} = \frac{2}{1}$

So $\vec{OP} = \frac{\vec{b} + 2\vec{a}}{3}$ and $\vec{OQ} = \frac{2\vec{b} + \vec{a}}{3}$

$$L.H.S = OP^2 + OQ^2$$

$$= \vec{OP}\cdot\vec{OP} + \vec{OQ}\cdot\vec{OQ}$$

$$= \left(\frac{\vec{b}+2\vec{a}}{3}\right)\cdot\left(\frac{\vec{b}+2\vec{a}}{3}\right) + \left(\frac{2\vec{b}+\vec{a}}{3}\right)\cdot\left(\frac{2\vec{b}+\vec{a}}{3}\right)$$

$$= \frac{1}{9}\left\{\vec{b}\cdot\vec{b} + 2\vec{b}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + 4\vec{a}\cdot\vec{a} + 4\vec{b}\cdot\vec{b} + 2\vec{b}\cdot\vec{a} + 2\vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{a}\right\}$$

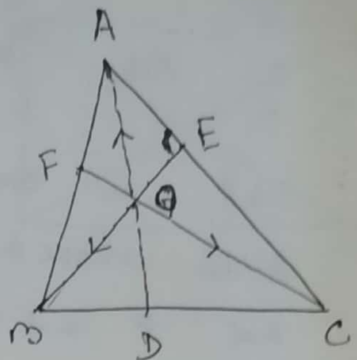
$$= \frac{5}{9}\left\{\vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b}\right\} \quad \{\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a} = 0$$

$$= \frac{5}{9}\left\{|\vec{a}|^2 + |\vec{b}|^2\right\}$$

$$= \frac{5}{9} AB^2 \quad \{\because AB^2 = |\vec{a}|^2 + |\vec{b}|^2\}$$

$$= R.H.S.$$

Examples. Prove that altitudes of a triangle are concurrent.



Proof: Consider a triangle ABC, whose altitudes BE and CF meet at the point O.

Now joining the pts A and O extending it towards the pt O so that it meets at the pt D.

We have to prove AD is ⊥r to BC.

Let $\vec{OA} = \vec{a}$, $\vec{OB} = \vec{b}$, $\vec{OC} = \vec{c}$, where 'O' is the origin.

$\therefore$ $\vec{AB} = \vec{OB} - \vec{OA} = \vec{b} - \vec{a}$

$\vec{CA} = \vec{OA} - \vec{OC} = \vec{a} - \vec{c}$

$\vec{CB} = \vec{OB} - \vec{OC} = \vec{b} - \vec{c}$

Since OB is ⊥r to CA and OC is ⊥r to AB, therefore

$(\vec{a} - \vec{c}) \cdot \vec{b} = 0$

and $(\vec{b} - \vec{a}) \cdot \vec{c} = 0$

Adding the above two eqs we get

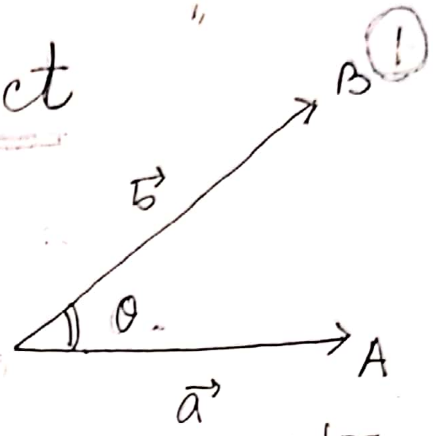
$(\vec{b} - \vec{c}) \cdot \vec{a} = 0$

$\therefore$ $\vec{OA}$ is ⊥r to $\vec{CB}$.

or, $\vec{AD}$ is ⊥r to $\vec{BC}$.

Hence the altitudes of a triangle are concurrent.

Cross Product



Def.

$\vec{a} \times \vec{b} = |\vec{a}|\,|\vec{b}| \sin\theta \,\hat{n}$

Where θ is the angle between the vectors $\vec{a}$ and $\vec{b}$ originating from a common point, and $\hat{n}$ is an unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

① If $\vec{a}$ is parallel to $\vec{b}$ then $\vec{a} \times \vec{b} = \vec{0}$

If $\vec{a} \times \vec{b} = \vec{0}$, then $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ is parallel to $\vec{b}$.

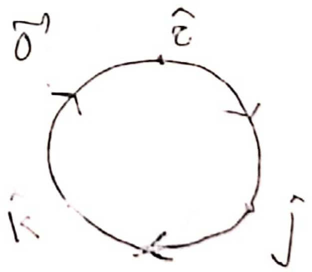
③ If $\vec{a}$ and $\vec{b}$ are <u>like</u> vectors then $\vec{a} \times \vec{b} = \vec{0}$ $\{\because \theta = 0°$

④ If $\vec{a}$ and $\vec{b}$ are unlike vectors then $\vec{a} \times \vec{b} = \vec{0}$ $\{\because \theta = 180°$

⑤ $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0}$

$\hat{i} \times \hat{j} = \hat{k}$

$\hat{j} \times \hat{i} = -\hat{k}$



⑥ $\vec{a} \times \vec{a} = \vec{0}$

$\vec{a} \cdot \vec{a} = |\vec{a}|^2 = a^2 = (\vec{a})^2$

Let

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

Now $\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$

$= (a_1b_1)\,\hat{i}\times\hat{i} + (a_1b_2)(\hat{i}\times\hat{j}) + (a_1b_3)(\hat{i}\times\hat{k})$

$+ (a_2b_1)(\hat{j}\times\hat{i}) + (a_2b_2)\,\hat{j}\times\hat{j} + (a_2b_3)(\hat{j}\times\hat{k})$

$+ (a_3b_1)(\hat{k}\times\hat{i}) + (a_3b_2)(\hat{k}\times\hat{j}) + (a_3b_3)(\hat{k}\times\hat{k})$

$= a_1b_2\hat{k} - a_1b_3\hat{j} - a_2b_1\hat{k} + a_2b_3\hat{i} + a_3b_1\hat{j} - a_3b_2\hat{i}$

$= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Properties

(1) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$

(2) Cross product is not associative but associative w.r.to a scalar.

$$\lambda(\vec{a}\times\vec{b}) = (\lambda\vec{a})\times\vec{b} = \vec{a}\times(\lambda\vec{b})$$

where $\lambda > 0$

(3) Distributive law.

$$\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b} + \vec{a}\times\vec{c}$$

prove

$$\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

(3)

Proof

Let $\vec{a}, \vec{b}, \vec{c} \in V_3$ where V_3 be the set of all three dimensional vectors.

Again let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$L.H.S = \vec{a} \times (\vec{b} + \vec{c})$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \{(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})\}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times \{(b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}\}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

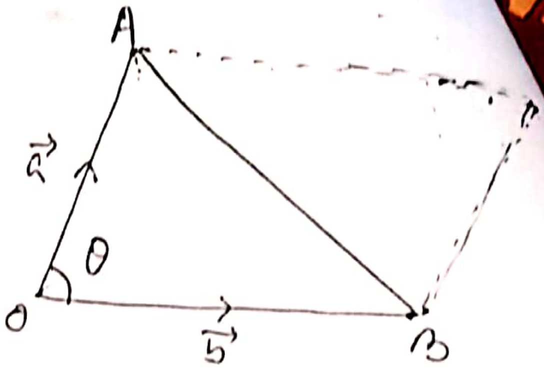
$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) + (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$$

$$= R.H.S.$$

✓



...ce of the Δle OAB

$= \frac{1}{2}(OB)(OA)\sin\theta$

$= \frac{1}{2}|\vec{OA}|\,|\vec{OB}|\sin\theta$

$= \frac{1}{2}|\vec{a}|\,|\vec{b}|\sin\theta$

or, $|\vec{a}|\,|\vec{b}|\sin\theta = 2$ (Area of Δle OAB)

= Area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

But $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}$ = Vector area.

Therefore, $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$

= Area of the parallelogram whose sides are $\vec{a}$ and $\vec{b}$.

Problems

(3)

Example:- (1) Determine the area of the parallelogram whose adjacent sides are $\hat{i}-3\hat{j}+\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$.

Solution:- Let $\vec{a} = \hat{i}-3\hat{j}+\hat{k}$

$\vec{b} = \hat{i}+\hat{j}+\hat{k}$

Now $\vec{a}\times\vec{b} = (\hat{i}-3\hat{j}+\hat{k})\times(\hat{i}+\hat{j}+\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= -4\hat{i}+4\hat{k}$$

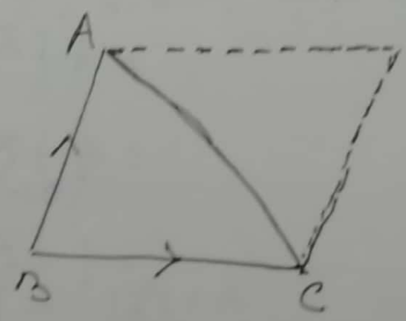
Hence the area of the parallelogram

$$= |\vec{a}\times\vec{b}|$$

$$= \sqrt{(-4)^2+4^2}$$

$$= 4\sqrt{2} \text{ sq. units.}$$

Example-(2) Find the area of the triangle whose vertices are $A(1,2,3)$ $B(-1,-1,0)$ and $C(1,-1,0)$



Solution:- Given the vertices of the are $A(1,2,3)$, $B(-1,-1,0)$ and $C(1,-1,0)$

i.e, $\vec{OA} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{OB} = -\hat{i}-\hat{j}-$

$\vec{OC} = \hat{i}\ldots$

✓

Now $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = 2\hat{i} + 3\hat{j} + 3\hat{k}$

and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = 2\hat{i}$

$\therefore \overrightarrow{BA} \times \overrightarrow{BC} = (2\hat{i} + 3\hat{j} + 3\hat{k}) \times 2\hat{i}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 3 \\ 2 & 0 & 0 \end{vmatrix}$$

$$= 6\hat{j} - 6\hat{k}$$

Area of the triangle $= \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}|$

$$= \frac{1}{2} \sqrt{6^2 + (-6)^2}$$

$$= 3\sqrt{2} \text{ sq. units.}$$

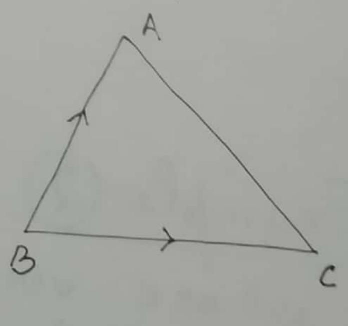
Example - 3

Show that the vector area of the triangle whose vertices have position vectors $\vec{a}, \vec{b}, \vec{c}$ is $\frac{1}{2}(\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a})$.

Solution: Let ABC be a triangle, whose vertices A, B, C have the position vectors $\vec{a}, \vec{b}, \vec{c}$ respectively.

Then $\overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \vec{a} - \vec{b}$

and $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \vec{c} - \vec{b}$



Now Vector area of the Δ^{le} ABC

$$= \frac{1}{2}\{\overrightarrow{BC} \times \overrightarrow{BA}\}$$

$$= \frac{1}{2}\{(\vec{c} - \vec{b}) \times (\vec{a} - \vec{b})\}$$

$$= \frac{1}{2}\{\vec{c} \times \vec{a} - \vec{c} \times \vec{b} - \vec{b} \times \vec{a} + \vec{b} \times \vec{b}\}$$

$$= \frac{1}{2}[\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}]$$

Given $\vec{a}+\vec{b} = 3\hat{i}+\hat{j}-2\hat{k}$

$\vec{a}-\vec{b} = \hat{i}-3\hat{j}+4\hat{k}$

Therefore $2\vec{a} = 4\hat{i}-2\hat{j}+2\hat{k}$

or, $\vec{a} = 2\hat{i}-\hat{j}+\hat{k}$

and $2\vec{b} = 2\hat{i}+4\hat{j}-6\hat{k}$

$\therefore \vec{b} = \hat{i}+2\hat{j}-3\hat{k}$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix}$

$= (3-2)\hat{i} + (1+6)\hat{j} + (4+1)\hat{k}$

$= \hat{i}+7\hat{j}+5\hat{k}$

Hence the area of the parallelogram

$= |\vec{a}\times\vec{b}|$

$= |\hat{i}+7\hat{j}+5\hat{k}|$

$= \sqrt{1^2+7^2+5^2}$

$= 5\sqrt{3}$ sq. units.

Example :- 6 Prove that $(\vec{a}\times\vec{b})^2 = a^2b^2 - (\vec{a}\cdot\vec{b})^2$

Proof L.H.S $= (\vec{a}\times\vec{b})^2$

$= \{|\vec{a}||\vec{b}|\sin\theta\,\hat{n}\}^2$ where $\hat{n}$ is an unit vector $\perp$r to $\vec{a}$ and $\vec{b}$

$= |\vec{a}|^2|\vec{b}|^2\sin^2\theta$ $[\because \hat{n}^2 = \hat{n}\cdot\hat{n} = 1$

$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= |\vec{a}|^2|\vec{b}|^2 - \{|\vec{a}||\vec{b}|\cos\theta\}^2$

$= a^2b^2 - (\vec{a}\cdot\vec{b})^2$ $\{\because |\vec{a}| = a$, $|\vec{b}| = b$.

$=$ R.H.S

5

7. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then show that $2\sin\theta/2 = |\hat{a} - \hat{b}|$

Proof

Given $\hat{a}$ and $\hat{b}$ are two unit vectors inclined at an angle θ.

$$\therefore \hat{a}\cdot\hat{b} = |\hat{a}|\,|\hat{b}|\cos\theta$$

$$\text{or, } \cos\theta = \hat{a}\cdot\hat{b}$$

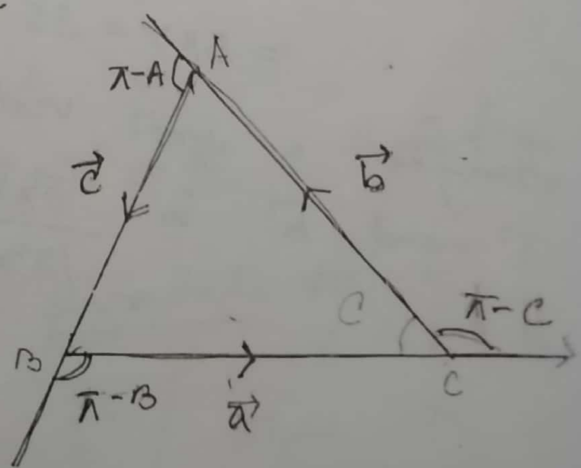
$$\therefore 2 - 2\cos\theta = \hat{a}\cdot\hat{a} + \hat{b}\cdot\hat{b} - 2\hat{a}\cdot\hat{b}$$

$$\text{or, } 2(1-\cos\theta) = (\hat{a}-\hat{b})^2$$

$$\text{or, } 2\times 2\sin^2\theta/2 = |\hat{a}-\hat{b}|^2$$

$$\text{or, } 2\sin\theta/2 = |\hat{a}-\hat{b}|$$

EX - 8. Prove that in any triangle $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ by vector method



Let us consider a triangle where $\vec{BC} = \vec{a}$, $\vec{CA} = \vec{b}$ and $\vec{AB} = \vec{c}$

Then the vector area of the $\triangle$le ABC is

$$\frac{1}{2}(\vec{a}\times\vec{b}) = \frac{1}{2}(\vec{b}\times\vec{c}) = \frac{1}{2}(\vec{c}\times\vec{a})$$

i.e, $|\vec{a}\times\vec{b}| = |\vec{b}\times\vec{c}| = |\vec{c}\times\vec{a}|$

$\therefore$ $ab\sin(\pi - C) = bc\sin(\pi - A) = ca\sin(\pi - B)$

where $|\vec{a}| = a$, $|\vec{b}| = b$, $|\vec{c}| = c$

i.e, $ab\sin C = bc\sin A = ca\sin B$

i.e, $\dfrac{\sin C}{c} = \dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

i.e, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.

Example (9) Find the unit vector perpendicular to the vector $\vec{a} = 4\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - 6\hat{j} - 3\hat{k}$

Given $\vec{a} = 4\hat{i} + 3\hat{j} - \hat{k}$

$\vec{b} = 2\hat{i} - 6\hat{j} - 3\hat{k}$

Now $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 3 & -1 \\ 2 & -6 & -3 \end{vmatrix}$

$= -15\hat{i} + 10\hat{j} - 30\hat{k}$

$\therefore$ $|\vec{a}\times\vec{b}| = \sqrt{(-15)^2 + 10^2 + (-30)^2}$

$= 5\times 7 = 35$

Therefore a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is $\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|} = \left(\dfrac{-3}{7}\right)\hat{i} + \left(\dfrac{2}{7}\right)\hat{j} + \left(\dfrac{-6}{7}\right)\hat{k}$

or

$\left(\dfrac{3}{7}\right)\hat{i} + \left(\dfrac{-2}{7}\right)\hat{j} + \left(\dfrac{6}{7}\right)\hat{k}$

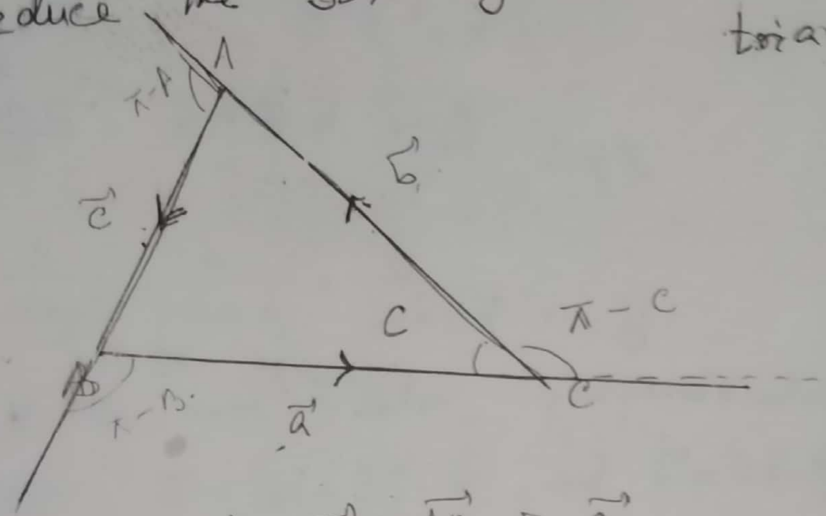
Example :- (11)

Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ represent the vectors $\vec{BC}$, $\vec{CA}$ and $\vec{AB}$ respectively. Show that

$$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.$$

and deduce the sine formula for a triangle.

Proof



Now

$$\vec{a}+\vec{b}+\vec{c}=\vec{BC}+\vec{CA}+\vec{AB}=\vec{0}$$

or, $\vec{a}+\vec{c}=-\vec{b}$

$$(\vec{a}+\vec{c})\times\vec{b}=-\vec{b}\times\vec{b}$$

or, $\vec{a}\times\vec{b}+\vec{c}\times\vec{b}=\vec{0}$

or $\vec{a}\times\vec{b}=-\vec{c}\times\vec{b}$

or $\boxed{\vec{a}\times\vec{b}=\vec{b}\times\vec{c}.}$ —— (1)

Similarly $\boxed{\vec{b}\times\vec{c}=\vec{c}\times\vec{a}}$ —— (2)

From (1) and (2) we get.

$$\boxed{\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}.}$$

So $|\vec{a}\times\vec{b}|=|\vec{b}\times\vec{c}|=|\vec{c}\times\vec{a}|$

$$|\vec{a}||\vec{b}|\sin(\pi-C)=|\vec{b}||\vec{c}|\sin(\pi-A)=|\vec{c}||\vec{a}|\sin(\pi-B)$$

or $ab\sin C=bc\sin A=ca\sin B$

Engg. Math-II.
2nd Sem-2020.
Question Set.

Time - 3 hrs.
F.M = 80.

Group-A

No-1 (i) Write the value of $(\hat{i}-\hat{j}+2\hat{k})\cdot(\hat{i}+\hat{j})$

(ii) What is the unit vector in the direction of the vector $3\hat{i}+4\hat{j}$.

(iii) Correct the error if any
A null vector has no direction.

(iv) Is $(\vec{a}\cdot\vec{b})\vec{c}$ a vector quantity.?.

(v) Find $\lim_{x\to 2} \frac{x^3-8}{x-2}$

(vi) Find $\frac{dy}{dx}$ if $y=\log\sqrt{x}$.

(vii) What is the value of $\lim_{x\to 0^+} \frac{|x|}{x}$.

(viii) If $f=e^{xy}$, Find $\frac{\partial f}{\partial x}$.

(ix) $\int \log x \, dx$.

(x) Write the order and degree of the following diffⁿ. Eqⁿ.

$$\sqrt{\left[1+\frac{dy}{dx}\right]} = \frac{d^2y}{dx^2}.$$

Group - B

Answer any six

No - 2(i) What is the scalar projection of $\hat{i}+\hat{j}-\hat{k}$ on $2\hat{i}-\hat{j}-\hat{k}$.

(ii) Find the area of the parallelogram whose adjacent sides are given by the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $2\hat{i}+\hat{j}-\hat{k}$.

(iii) Find $\lim_{n\to\infty} \frac{\lfloor n}{\lfloor (n+1) - \lfloor n}$.

(iv) Find $\frac{\partial f}{\partial x}$ where $f = \sin^{-1}(x/y)$

(v) Find $\frac{dy}{dx}$ if $x^y = y^x$.

(vi) $\int e^{\cos^2 x} \cdot \sin 2x \, dx$

(vii) $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx$

Group - C

Answer any three

No-3(a) Find the value of x, so that the vectors $\vec{a} = x\hat{i} - 3\hat{j} + 5\hat{k}$ and $\vec{b} = -x\hat{i} + x\hat{j} + 2\hat{k}$.

(b) $\lim_{n \to \infty} \frac{1^2 + 2^2 + 3^2 + \cdots + n^2}{n^4}$.

No-4(a) Find $\frac{dy}{dx}$ if $\ln\sqrt{x^2+y^2} = \tan^{-1}(y/x)$.

(b) Solve:- $\frac{dy}{dx} + y/x = x^2$.

No-5(a) $\int \frac{dx}{x[1+(\log x)^2]}$.

(b) Prove that $\int_0^{\pi/2} \log(\tan x)dx = 0$.

No-6(a) Find $\frac{dy}{dx}$ if $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$.

(b) $\int \frac{dx}{x^2 + a^2}$.

No-7(a) Prove that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$.

(b) $\int \frac{x^6}{x^2+1} dx$.

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Ali Sir.