

LECTURE NOTESON

CONTROL SYSTEM ENGINEERING(Theory-4)

6th Semester

(Prepared By Satyajit Pani,Lect. Electrical)



**DEPARTMENT OF ELECTRICAL ENGINEERING
ITT CHOUWAR**

BASIC DEFINITION IN BLOCK DIAGRAM MODEL:

Block diagram: It is the pictorial representation of the cause-and-response relationship between input and output of a physical system.

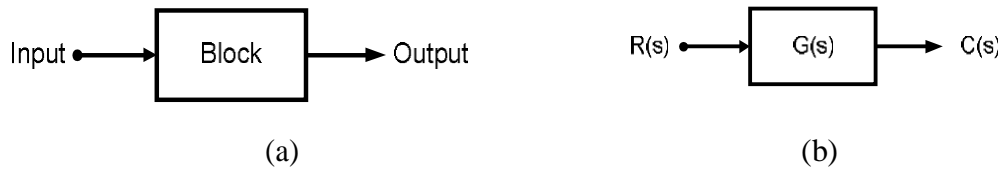


Fig. (a) A block diagram representation of a system and
(b) A block diagram representation with gain of a system

Output: The value of input multiplied by the gain of the system.

$$C(s) = G(s)R(s)$$

Summing point: It is the component of a block diagram model at which two or more signals can be added or subtracted. In Fig, inputs $R(s)$ and $B(s)$ have been given to a summing point and its output signal is $E(s)$. Here,

$$E(s) = R(s) - B(s)$$

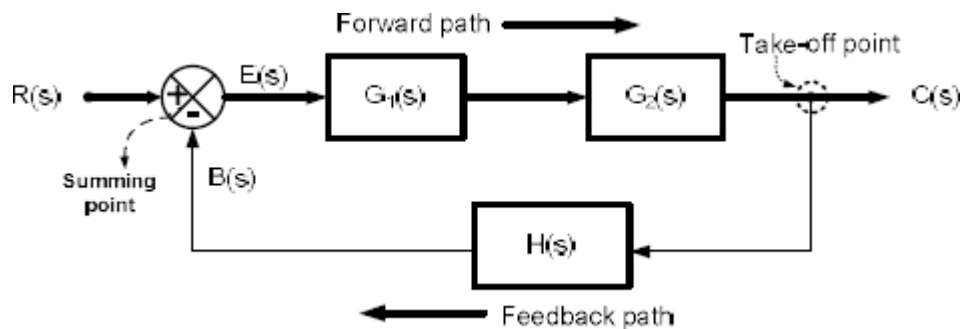


Fig. A block diagram representation of a system showing its different components

Take-off point: It is the component of a block diagram model at which a signal can be taken directly and supplied to one or more points as shown in Fig.

Forward path: It is the direction of signal flow from input towards output.

Feedback path: It is the direction of signal flow from output towards input.

RULES FOR REDUCTION OF BLOCK DIAGRAM MODEL:

Sl No.	Rule No.	Configuration	Equivalent	Name
1	Rule 1			Cascade
2	Rule 2			Parallel
3	Rule 3			Loop
4	Rule 4			Associative Law
5	Rule 5			Move take-off point after a block
6	Rule 6			Move take-off point before a block
7	Rule 7			Move summing-point after a block
8	Rule 8			Move summing-point before a block
9	Rule 9			Move take-off point after a summing-point
10	Rule 10			Move take-off point before a summing-point

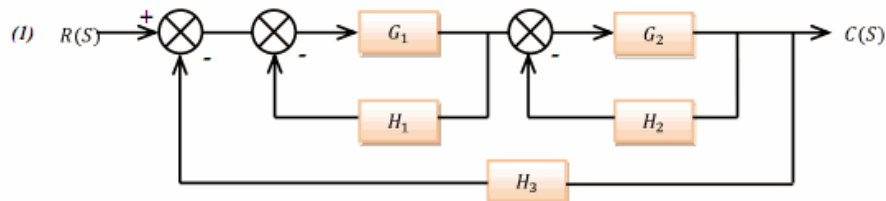
PROCEDURE FOR REDUCTION OF BLOCK DIAGRAM MODEL:

- Step 1:** Reduce the cascade blocks.
- Step 2:** Reduce the parallel blocks.
- Step 3:** Reduce the internal feedback loops.
- Step 4:** Shift take-off points towards right and summing points towards left.
- Step 5:** Repeat step 1 to step 4 until the simple form is obtained.
- Step 6:** Find transfer function of whole system as $\frac{C(s)}{R(s)}$.

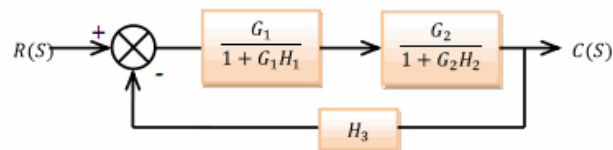
PROCEDURE FOR FINDING OUTPUT OF BLOCK DIAGRAM MODEL WITH MULTIPLE INPUTS:

- Step 1:** Consider one input taking rest of the inputs zero, find output.
- Step 2:** Follow step 1 for each inputs of the given Block Diagram model and find their corresponding outputs.
- Step 3:** Find the resultant output by adding all individual outputs.

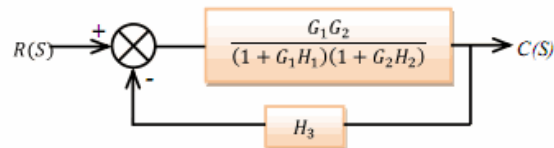
Example 1:- Find C(s)/R(s) using block diagram reduction rules



Solution: By eliminating the feed-back paths, we get



Combining the blocks in series, we get



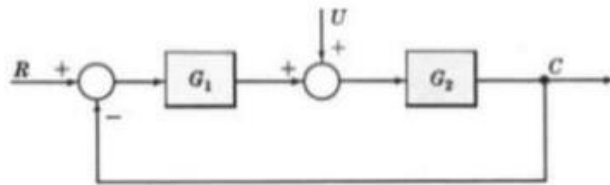
Eliminating the feed back path, we get

$$R(S) \rightarrow \frac{\frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2)}}{1 + \frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2)} \cdot H_3} \rightarrow C(S)$$

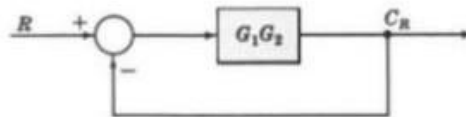
$$R(S) \rightarrow \frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2) + G_1 G_2 H_3} \rightarrow C(S)$$

$$\Rightarrow TF = \frac{C(S)}{R(S)} = \frac{G_1 G_2}{(1 + G_1 H_1)(1 + G_2 H_2) + G_1 G_2 H_3}$$

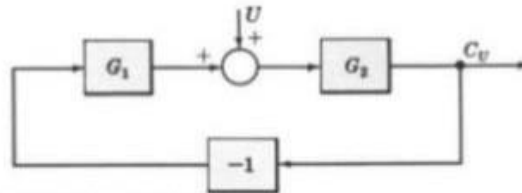
Example-2:- Determine output C due to inputs R & U using the superposition method.



- Step 1:** Put $U = 0$.
Step 2: The system reduces to

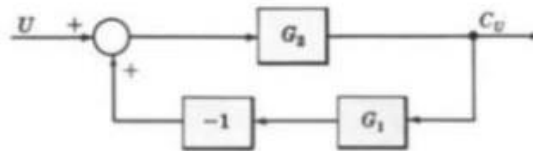


- Step 3:** the output C_R due to input R is $C_R = [G_1G_2/(1 + G_1G_2)]R$.

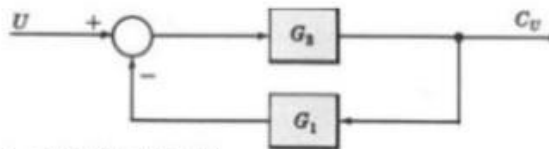


- Step 4a:** Put $R = 0$.
Step 4b: Put -1 into a block, representing the negative feedback effect:

Rearrange the block diagram:



Let the -1 block be absorbed into the summing point:



- Step 4c:** the output C_U due to input U is $C_U = [G_2/(1 + G_1G_2)]U$.

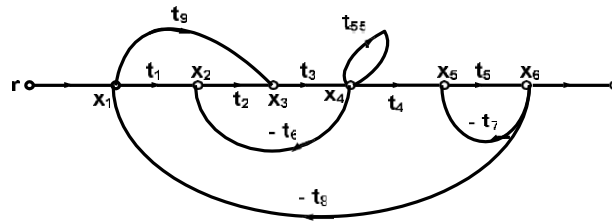
- Step 5:** The total output is $C = C_R + C_U$

$$= \left[\frac{G_1G_2}{1 + G_1G_2} \right] R + \left[\frac{G_2}{1 + G_1G_2} \right] U$$

$$= \left[\frac{G_2}{1 + G_1G_2} \right] [G_1R + U]$$

SIGNAL FLOW GRAPHS (SFGS)

It is a pictorial representation of a system that graphically displays the signal transmission in it.



Basic Definitions in SFGs:

Input or source node: It is a node that has only outgoing branches i.e. node 'r'.

Output or sink node: It is a node that has only incoming branches i.e. node 'c'.

Chain node: It is a node that has both incoming and outgoing branches i.e. nodes 'x1', 'x2', 'x3', 'x4', 'x5' and 'x6'.

Gain or transmittance: It is the relationship between variables denoted by two nodes or value of a branch., Transmittances are 't1', 't2', 't3', 't4', 't5' and 't6'.

Forward path: It is a path from input node to output node without repeating any of the nodes in between them. There are two forward paths, i.e. path-1: 'r-x1-x2-x3-x4-x5-x6-c' and path-2: 'r-x1-x3-x4-x5-x6-c'.

Feedback path: It is a path from output node or a node near output node to a node near input node without repeating any of the nodes in between them.

Loop: It is a closed path that starts from one node and reaches the same node after trading through other nodes. There are four loops, i.e. loop-1: 'x2-x3-x4-x1', loop-2: 'x5-x6-x5', loop-3: 'x1-x2-x3-x4-x5-x6-x1' and loop-4: 'x1-x3-x4-x5-x6-x1'.

Self Loop: It is a loop that starts from one node and reaches the same node without trading through other nodes i.e. loop in node 'x4' with transmittance 't55'.

Path gain: It is the product of gains or transmittances of all branches of a forward path. The path gains are $P_1 = t_1 t_2 t_3 t_4 t_5$ (for path-1) and $P_2 = t_9 t_3 t_4 t_5$ (for path-2).

Loop gain: It is the product of gains or transmittances of all branches of a loop. There are four loops, i.e. $L_1 = -t_2 t_3 t_6$, $L_2 = -t_5 t_7$, $L_3 = -t_1 t_2 t_3 t_4 t_5 t_8$, and $L_4 = -t_9 t_3 t_4 t_5 t_8$.

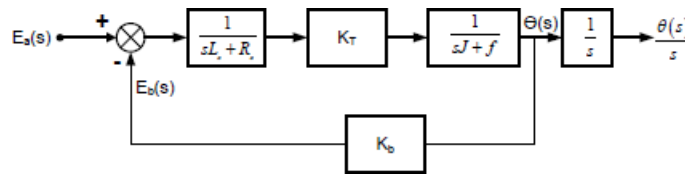
Dummy node: If the first node is not an input node and/or the last node is not an output node than a node is connected before the existing first node and a node is connected after the existing last node with unity transmittances. These nodes are called dummy nodes. 'r' and 'c' are the dummy nodes.

Non-touching Loops: Two or more loops are non-touching loops if they don't have any common nodes between them. L_1 and L_2 are non-touching loops

PROPERTIES OF SFGS:

- Applied to linear system
- Arrow indicates signal flow
- Nodes represent variables, summing points and take-off points
- Algebraic sum of all incoming signals and outgoing nodes is zero
- SFG of a system is not unique
- Overall gain of an SFG can be determined by using Mason's gain formula

SFG FROM BLOCK DIAGRAM MODEL:

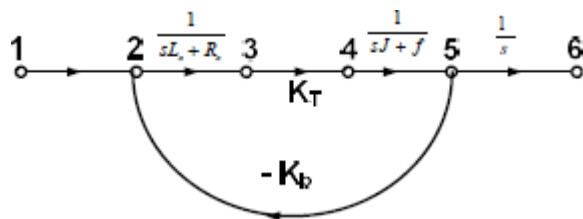
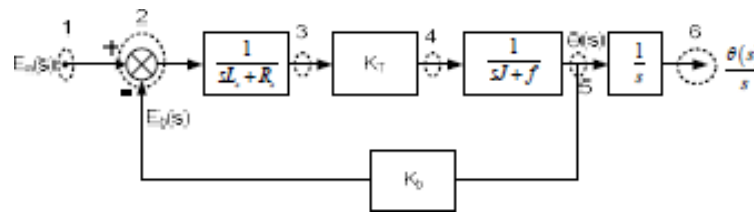


Step-1: All variables and signals are replaced by nodes.

Step-2: Connect all nodes according to their signal flow.

Step-3: Each of gains is replaced by transmittances of the branches connected between two nodes of the forward paths.

Step-4: Each of gains is replaced by transmittances multiplied with (-1) of the branches connected between two nodes of the forward paths.



MASON'S GAIN FORMULA:

Transfer function of a system=

$$G(s) = \frac{C(s)}{R(s)} = \frac{\sum_{k=1}^N P_k \Delta_k}{\Delta}$$

Where,

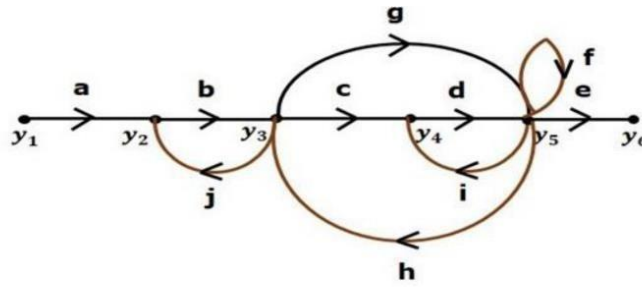
N= total number of forwardpaths

P_k= path gain of kth forward path

Δ= 1 - (Σloop gains of all individual loops) + (Σgain product of loop gains of all possible two non-touching loops) - (Σgain product of loop gains of all possible three non-touching loops) + ...

Δ_k= value of Δ after eliminating all loops that touches kth forward path

Example:- Find overall transfer function of system using Mason's gain formula



- ▣ Number of forward paths, $N = 2$.
- ▣ First forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_6$.
- ▣ First forward path gain, $p_1 = abcde$.
- ▣ Second forward path is - $y_1 \rightarrow y_2 \rightarrow y_3 \rightarrow y_5 \rightarrow y_6$.
- ▣ Second forward path gain, $p_2 = abge$.
- ▣ Number of individual loops, $L = 5$.
- ▣ Loops are - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_3 \rightarrow y_5 \rightarrow y_3$, $y_3 \rightarrow y_4 \rightarrow y_5 \rightarrow y_3$, $y_4 \rightarrow y_5 \rightarrow y_4$ and $y_5 \rightarrow y_5$.
- ▣ Loop gains are - $l_1 = bj$, $l_2 = gh$, $l_3 = cdh$, $l_4 = di$ and $l_5 = f$.
- ▣ Number of two non-touching loops = 2.
- ▣ First non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_4 \rightarrow y_5 \rightarrow y_4$.
- ▣ Gain product of first non-touching loops pair, $l_1 l_4 = bjdi$
- ▣ Second non-touching loops pair is - $y_2 \rightarrow y_3 \rightarrow y_2$, $y_5 \rightarrow y_5$.
- ▣ Gain product of second non-touching loops pair is - $l_1 l_5 = bjf$

Higher number of (more than two) non-touching loops are not present in this signal flow graph.

We know,

$$\Delta = 1 - (\text{sum of all individual loop gains}) + (\text{sum of gain products of all possible two nontouching loops}) - (\text{sum of gain products of all possible three nontouching loops}) + \dots$$

Substitute the values in the above equation,

$$\Delta = 1 - (bj + gh + cdh + di + f) + (bjdi + bjf) - (0)$$

$$\Rightarrow \Delta = 1 - (bj + gh + cdh + di + f) + bjdi + bjf$$

There is no loop which is non-touching to the first forward path.

So, $\Delta_1 = 1$.

Similarly, $\Delta_2 = 1$. Since, no loop which is non-touching to the second forward path.

Substitute, $N = 2$ in Mason's gain formula

$$T = \frac{C(s)}{R(s)} = \frac{\sum_{i=1}^2 P_i \Delta_i}{\Delta}$$

$$T = \frac{C(s)}{R(s)} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

Substitute all the necessary values in the above equation.

$$T = \frac{C(s)}{R(s)} = \frac{(abcde)1 + (abge)1}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

$$\Rightarrow T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

Therefore, the transfer function is -

$$T = \frac{C(s)}{R(s)} = \frac{(abcde) + (abge)}{1 - (bj + gh + cdh + di + f) + bjdi + bjf}$$

CONSTRUCTION OF SIGNAL FLOW GRAPH FROM ALGEBRAIC EQUATIONS:-

Let us construct a signal flow graph by considering the following algebraic equations

$$y_2 = a_{12}y_1 + a_{42}y_4$$

$$y_3 = a_{23}y_2 + a_{53}y_5$$

$$y_4 = a_{34}y_3$$

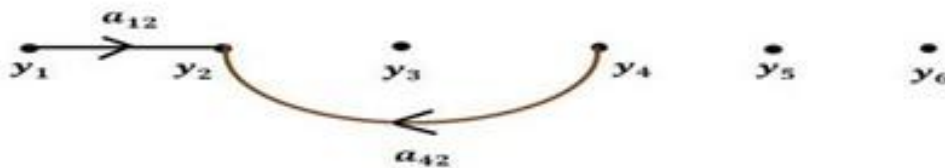
$$y_5 = a_{45}y_4 + a_{35}y_3$$

$$y_6 = a_{56}y_5$$

There will be six **nodes** (y_1, y_2, y_3, y_4, y_5 and y_6) and eight **branches** in this signal flow graph. The gains of the branches are $a_{12}, a_{23}, a_{34}, a_{45}, a_{56}, a_{42}, a_{53}$ and a_{35} .

To get the overall signal flow graph, draw the signal flow graph for each equation, then combine all these signal flow graphs and then follow the steps given below –

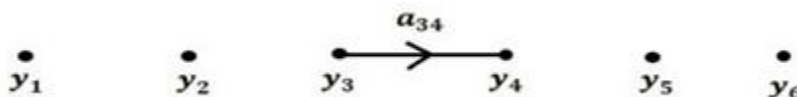
Step 1 – Signal flow graph for $y_2 = a_{12}y_1 + a_{42}y_4$ is shown in the following figure.



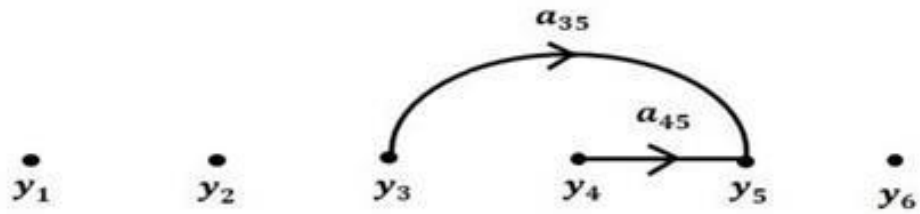
Step 2 – Signal flow graph for $y_3 = a_{23}y_2 + a_{53}y_5$ is shown in the following figure.



Step 3 – Signal flow graph for $y_4 = a_{34}y_3$ is shown in the following figure.



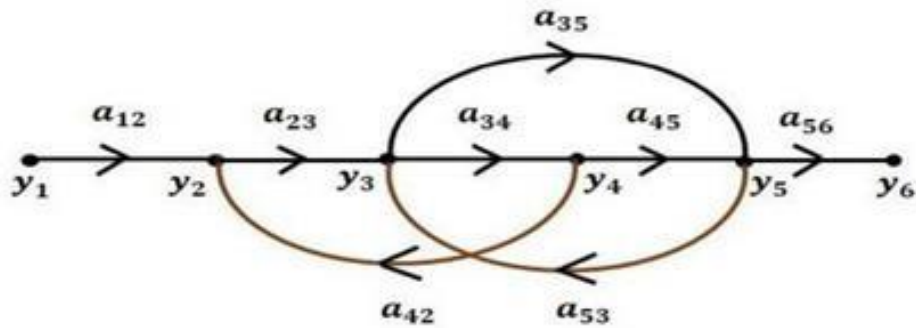
Step 4 – Signal flow graph for $y_5 = a_{45}y_4 + a_{35}y_3$ is shown in the following figure.



Step 5 – Signal flow graph for $y_6 = a_{56}y_5$ is shown in the following figure.



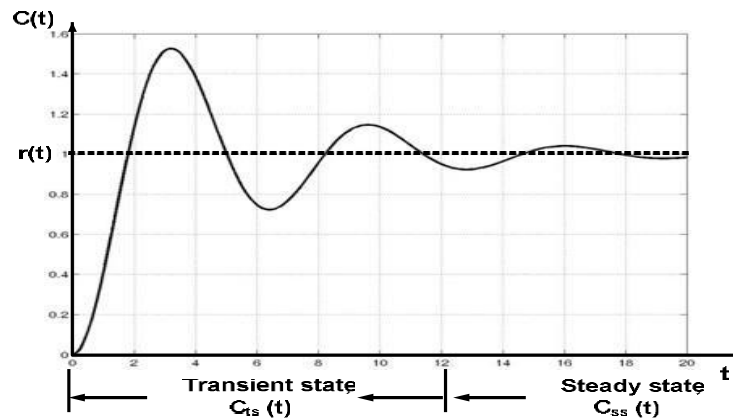
Step 6 – Signal flow graph of overall system is shown in the following figure.



CHAPTER#2 TIME RESPONSE ANALYSIS

TIME RESPONSE OF CONTROL SYSTEM:

Time response $c(t)$ is the variation of output with respect to time. The part of time response that goes to zero after large interval of time is called transient response $c_{tr}(t)$. The part of time response that remains after transient response is called steady-state response $c_{ss}(t)$.



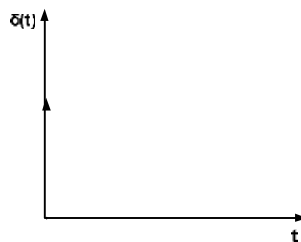
STANDARD TEST SIGNALS

1. **Impulse Signal:** An impulse signal $\delta(t)$ is mathematically defined as follows.

$$\delta(t) = \left. \begin{array}{l} \text{undefined} \\ 0 \end{array} \right\} \begin{array}{l} ; t = 0 \\ ; t \neq 0 \end{array}$$

Laplace transform of impulse signal is

$$\delta(s) = 1$$

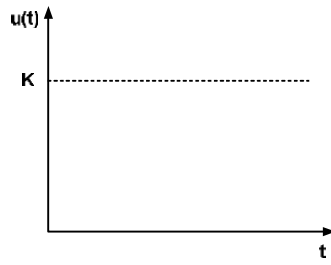


2. **Step Signal:** A step signal $u(t)$ is mathematically defined as follows.

$$u(t) = \left. \begin{array}{l} 0 \\ K \end{array} \right\} \begin{array}{l} ; t < 0 \\ ; t \geq 0 \end{array}$$

Laplace transform of step signal is

$$U(s) = \frac{K}{s}$$

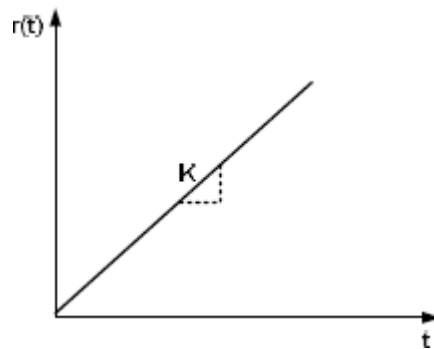


3. Ramp Signal: A step signal $r(t)$ is mathematically defined as follows.

$$(8.10) \quad r(t) = \left. \begin{array}{l} 0 \quad ; t < 0 \\ Kt \quad ; t \geq 0 \end{array} \right\}$$

Laplace transform of ramp signal

$$R(s) = \frac{K}{s^2}$$

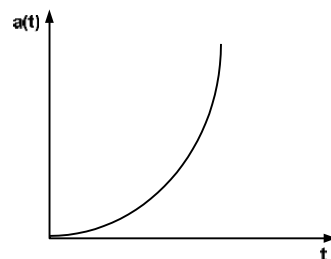


4. Parabolic Signal A step signal $a(t)$ is mathematically defined as follows.

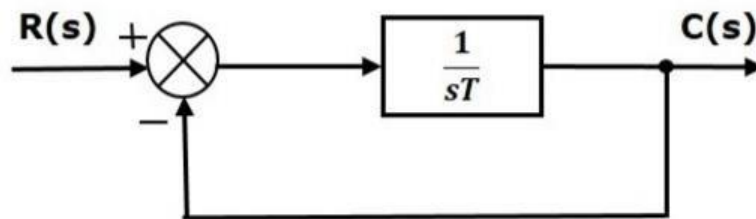
$$a(t) = \left. \begin{array}{l} 0 \quad ; t < 0 \\ \frac{Kt^2}{2} \quad ; t \geq 0 \end{array} \right\}$$

Laplace transform of parabolic signal is

$$A(s) = \frac{K}{s^3}$$



TIME RESPONSE OF 1ST ORDER SYSTEM:



We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT + 1} \right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal $c(t)$,
- **R(s)** is the Laplace transform of the input signal $r(t)$, and
- **T** is the time constant.

Follow these steps to get the response (output) of the first order system in the time domain.

- Take the Laplace transform of the input signal $r(t)$.
- Consider the equation, $C(s) = \left(\frac{1}{sT+1} \right) R(s)$

- Substitute $R(s)$ value in the above equation.
- Do partial fractions of $C(s)$ if required.
- Apply inverse Laplace transform to $C(s)$.

(i) Unit Step Response:-

Consider the **unit step signal** as an input to first order system.

So, $r(t)=u(t)$

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right) \left(\frac{1}{s}\right) = \frac{1}{s(sT+1)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{1}{s(sT+1)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s(sT+1)} = \frac{A(sT+1) + Bs}{s(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1=A(sT+1)+Bs$$

By equating the constant terms on both the sides, you will get $A = 1$.

Substitute, $A = 1$ and equate the coefficient of the s terms on both the sides.

$$0=T+B$$

$$\Rightarrow B=-T$$

Substitute, $A = 1$ and $B = -T$ in partial fraction expansion of $C(s)$

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s + \frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

The **unit step response**, $c(t)$ has both the transient and the steady state terms.

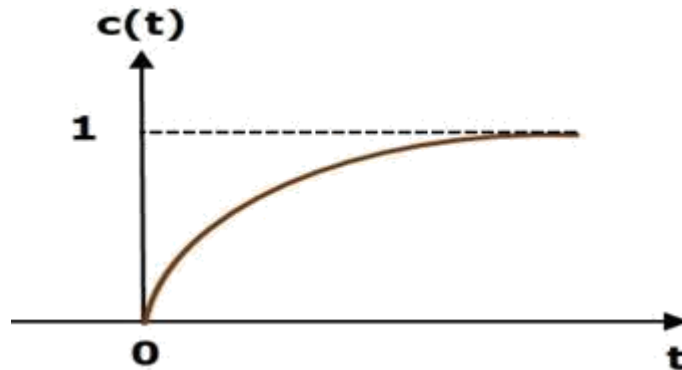
The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)} u(t)$$

The steady state term in the unit step response is -

$$c_{ss}(t) = u(t)$$

The following figure shows the unit step response



The value of the **unit step response**, $c(t)$ is zero at $t = 0$ and for all negative values of t . It is gradually increasing from zero value and finally reaches to one in steady state. So, the steady state value depends on the magnitude of the input.

(ii) Unit impulse response:

Consider the **unit impulse signal** as an input to the first order system.

$$\text{So, } r(t) = \delta(t)$$

Apply Laplace transform on both the sides.

$$R(s) = 1$$

$$\text{Consider the equation, } C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right) (1) = \frac{1}{sT+1}$$

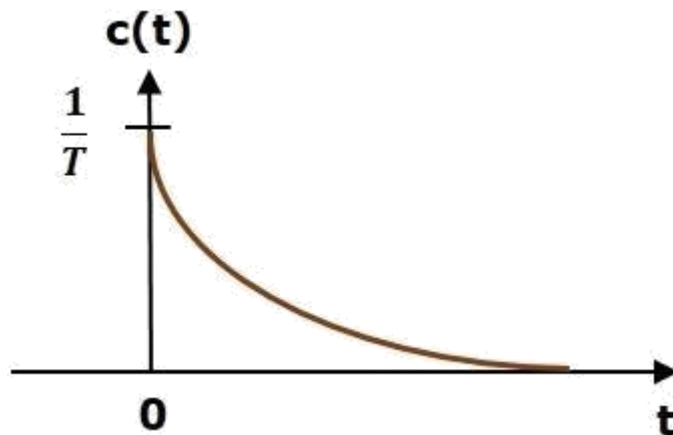
Rearrange the above equation in one of the standard forms of Laplace transforms.

$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T} \left(\frac{1}{s + \frac{1}{T}}\right)$$

Applying Inverse Laplace Transform on both the sides,

$$c(t) = \frac{1}{T} e^{-\left(\frac{t}{T}\right)} u(t)$$

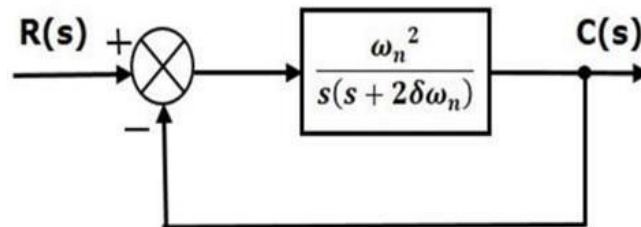
The unit impulse response is shown in the following figure.



The **unit impulse response**, $c(t)$ is an exponential decaying signal for positive values of 't' and it is zero for negative values of 't'.

TIME RESPONSE OF 2ND ORDER SYSTEM

Consider the following block diagram of closed loop control system. Here, an open loop transfer function, $\omega_n^2 / s(s+2\delta\omega_n)$ is connected with a unity negative feedback.



We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

STEP RESPONSE OF 2ND ORDER SYSTEM

Consider the unit step signal as an input to the second order system. Laplace transform of the unit step signal is,

$$\begin{aligned}T(s) &= \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ \text{Now, } r(t) &= 1 \text{ or } R(s) = \frac{1}{s} \\ \therefore C(s) &= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2} \\ &= \frac{1}{s} \cdot \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)} \\ &= \frac{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) - (s + \zeta\omega_n)^2 + \omega_n^2\zeta^2}{s \{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)\}} \\ &= \frac{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) - s^2 - 2s\zeta\omega_n - \omega_n^2\zeta^2 + \omega_n^2\zeta^2}{s \{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)\}} \\ &= \frac{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2) - s(s + 2s\zeta\omega_n)}{s \{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)\}} \\ &= \frac{1}{s} - \frac{s + 2s\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}\end{aligned}$$

Putting, $\omega_d = \omega_n\sqrt{1 - \zeta^2}$

$$\begin{aligned}&= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\end{aligned}$$

Taking the inverse Laplace Transform of above equation, we get

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2}\right]$$

$$= \mathcal{L}^{-1} \left[\frac{1}{s} \right] - \mathcal{L}^{-1} \left[\frac{s + \zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_d^2} \right] - \mathcal{L}^{-1} \left[\frac{\zeta\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s + \zeta\omega_n)^2 + \omega_d^2} \right]$$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t} \cdot \cos \omega_d t - \frac{\zeta\omega_n}{\omega_d} \cdot e^{-\zeta\omega_n t} \cdot \sin \omega_d t$$

$$\therefore \mathcal{L}^{-1} \left[\frac{1}{s} \right] = 1, \quad \mathcal{L}^{-1} \left[\frac{s + \alpha}{(s + \alpha)^2 + \omega^2} \right] = e^{-\alpha t} \cos \omega t,$$

$$\mathcal{L}^{-1} \left[\frac{\omega}{(s + \alpha)^2 + \omega^2} \right] = e^{-\alpha t} \sin \omega t$$

The above expression of output $c(t)$ can be rewritten as

$$c(t) = 1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta}{\sqrt{1 - \zeta^2}} \cdot \sin \omega_d t \right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \left(\sqrt{1 - \zeta^2} \cos \omega_d t + \zeta \cdot \sin \omega_d t \right)$$

$$\left[\text{Say, } \zeta = \cos \phi, \text{ hence, } \sqrt{1 - \zeta^2} = \sin \phi \right]$$

$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} (\sin \phi \cos \omega_d t + \cos \phi \sin \omega_d t)$$

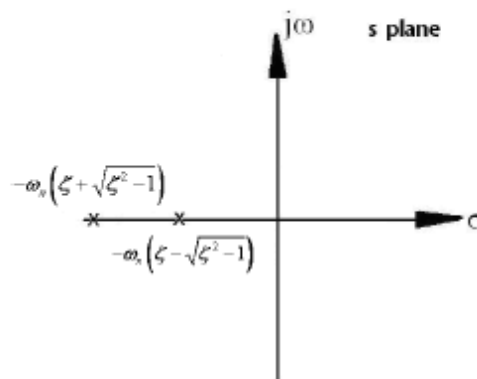
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin (\omega_d t + \phi)$$

(a) $\zeta > 1$ over damped

Poles are:

$$s_{1,2} = -\omega_n \left(\zeta \pm \sqrt{\zeta^2 - 1} \right)$$

Graphically, the poles of an over damped system is shown as follows.

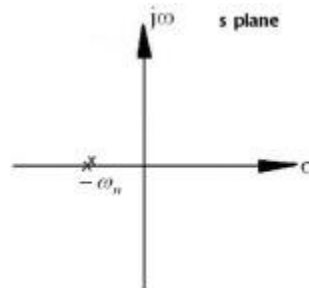


(b) $\zeta = 1$ critically damped

Poles are:

$$s_{1,2} = -\omega_n$$

Graphically, the poles of an critically damped system is shown as follows.



(c) $\zeta < 1$ under damped

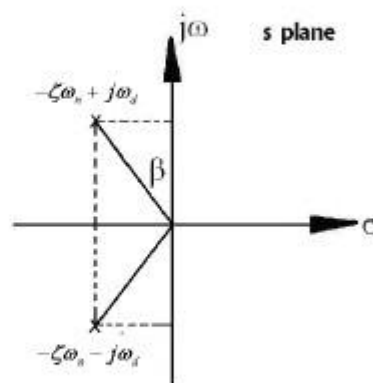
Poles are:

$$s_{1,2} = -\omega_n \left(\zeta \pm j\sqrt{1-\zeta^2} \right)$$
$$\Rightarrow s_{1,2} = -\zeta\omega_n \pm j\omega_d$$

Where, ω_d = Damped natural frequency

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

Graphically, the poles of an critically damped system is shown as follows.

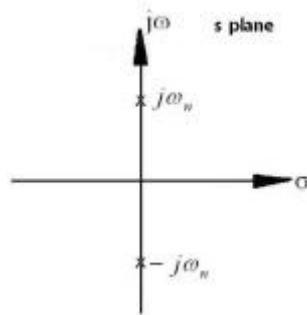


Here, $\tan \beta = \frac{\zeta}{\sqrt{1-\zeta^2}}$

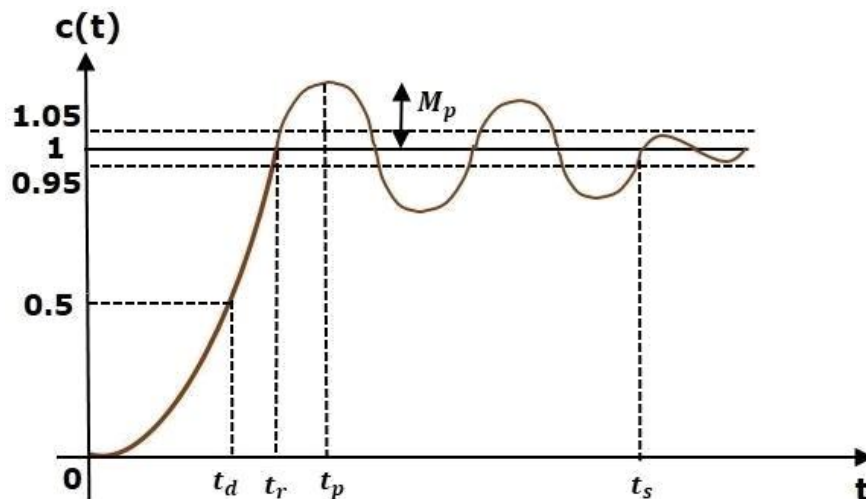
(d) $\zeta = 0$ un-damped

Poles are:

$$s_{1,2} = -\pm j\omega_n$$



TIME RESPONSE SPECIFICATION



Delay Time

It is the time required for the response to reach **half of its final value** from the zero instant. It is denoted by t_d .

Rise Time

It is the time required for the response to rise from **0% to 100% of its final value**. This is applicable for the **under-damped systems**. For the over-damped systems, consider the duration from 10% to 90% of the final value. Rise time is denoted by t_r .

As per definition, the magnitude of output signal at Rise times is 1. That is $c(t) = 1$, hence

$$\begin{aligned}
 1 &= 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t_r + \phi \right\} \\
 \Rightarrow \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t_r + \phi \right\} &= 0 \\
 \Rightarrow \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t_r + \phi \right\} &= 0 \\
 \Rightarrow \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t_r + \phi \right\} &= \pi \\
 \Rightarrow t_r &= \frac{\pi - \phi}{\omega_n \sqrt{1-\zeta^2}}
 \end{aligned}$$

Peak Time

It is the time required for the response to reach the **peak value** for the first time. It is denoted by t_p . At $t=t_p$ the first derivative of the response is zero.

As per definition at the peak time, the response curve reaches to its maximum value. Hence at that point,

$$\frac{dc(t)}{dt} = 0$$

$$\begin{aligned} \text{Now, } c(t) &= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} \\ \therefore \frac{dc(t)}{dt} &= -\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cdot \omega_n \sqrt{1-\zeta^2} \cos \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} \\ &\quad - \frac{(-\zeta\omega_n) e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} \\ \text{Putting, } \frac{dc(t)}{dt} &= 0 \\ \therefore \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} &\left[-\omega_n \sqrt{1-\zeta^2} \cos \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} + \zeta\omega_n \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} \right] \\ &= 0 \end{aligned}$$

$$\omega_n \sqrt{1-\zeta^2} \cos \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\} = \zeta\omega_n \sin \left\{ \left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right\}$$

$$\Rightarrow \tan \left[\left(\omega_n \sqrt{1-\zeta^2} \right) t + \phi \right] = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan \phi$$

$$\therefore \left(\omega_n \sqrt{1-\zeta^2} \right) t = n\pi$$

Where, $n = 1, 2, 3 \dots$

$$t_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

Peak Overshoot

Peak overshoot **M_p** is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

Mathematically, we can write it as

$$M_p = c(t_p) - c(\infty)$$

Where, $c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of the response.

$$\begin{aligned}
c(t)_{max} &= 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} \sin \left[\left(\omega_n \sqrt{1-\zeta^2} \right) t_p + \phi \right] \\
\Rightarrow c(t)_{max} &= 1 - \frac{e^{-\zeta\omega_n \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin \left[\left(\omega_n \sqrt{1-\zeta^2} \right) \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} + \phi \right] \\
\Rightarrow c(t)_{max} &= 1 - \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin (\pi + \phi) = 1 - \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} (-\sin \phi) \\
&= 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin \phi = 1 + \frac{e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sqrt{1-\zeta^2} = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}
\end{aligned}$$

$$\begin{aligned}
M_p &= c(t)_{max} - 1 = \left(1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \right) - 1 \\
\Rightarrow M_p &= e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}
\end{aligned}$$

Settling time

It is the time required for the response to reach the steady state and stay within the specified tolerance bands around the final value. In general, the tolerance bands are 2% and 5%. The settling time is denoted by t_s .

The settling time for 5% tolerance band is –

$$\text{The settling time for 2% tolerance band is } t_s = \frac{3}{\delta\omega_n} = 3\tau$$

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $1/\delta\omega_n$.

- Both the settling time t_s and the time constant τ are inversely proportional to the damping ratio δ .

Both the settling time t_s and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time t_s and time constant τ will never change.

Steady state error:-

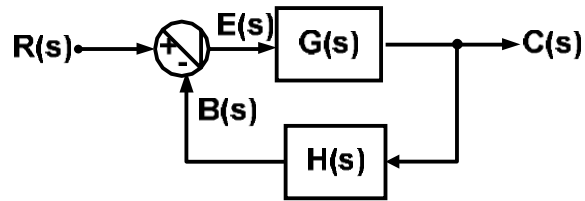
The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss} . We can find steady state error using the final value theorem as follows.

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} E(s)$$

Where,

$E(s)$ is the Laplace transform of the error signal, $e(t)$

A simple closed-loop control system with negative feedback is shown as follows.



$$E(s) = R(s) - B(s) \text{ --- (i)}$$

$$B(s) = C(s) H(s) \text{ ----(ii)}$$

$$C(s) = E(s) G(s) \text{ ---- (iii)}$$

Applying value of $B(s)$ of eq 2 into eq 1

$$E(s) = R(s) - C(s) H(s)$$

Applying value of $C(s)$ of eq 3 into above eq

$$E(s) = R(s) - E(s) G(s) H(s)$$

$$\Rightarrow E(s) = \frac{R(s)}{1 + G(s) H(s)}$$

Steady-state error,

$$e_{ss} = \lim_{t \rightarrow \infty} e(t) = \lim_{s \rightarrow 0} sE(s)$$

$$e_{ss} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{sR(s)}{1 + G(s) H(s)}$$

Therefore, steady-state error depends on two factors, i.e.

- (a) type and magnitude of $R(s)$
- (b) open-loop transfer function $G(s)H(s)$

Types of input and Steady-state error:

(i) Step Input:

$$R(s) = \frac{A}{s}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \left(\frac{A}{s} \right)}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{A}{1 + G(s)H(s)}$$

$$\Rightarrow e_{ss} = \frac{A}{1 + \lim_{s \rightarrow 0} G(s)H(s)} = \frac{A}{1 + K_p}$$

Where,

$$K_p = \lim_{s \rightarrow 0} G(s)H(s)$$

(ii) Ramp Input:

$$R(s) = \frac{A}{s^2}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \left(\frac{A}{s^2} \right)}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{A}{s [1 + G(s)H(s)]}$$

$$\Rightarrow e_{ss} = \lim_{s \rightarrow 0} \frac{A}{s + sG(s)H(s)}$$

$$\Rightarrow e_{ss} = \frac{A}{\lim_{s \rightarrow 0} sG(s)H(s)} = \frac{A}{K_v}$$

Where,

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$$

(iii) Parabolic Input:

$$R(s) = \frac{A}{s^3}$$

$$e_{ss} = \lim_{s \rightarrow 0} \frac{s \left(\frac{A}{s^3} \right)}{1 + G(s)H(s)} = \lim_{s \rightarrow 0} \frac{A}{s^2 [1 + G(s)H(s)]}$$

$$\Rightarrow e_{ss} = \lim_{s \rightarrow 0} \frac{A}{s^2 + s^2G(s)H(s)}$$

$$\Rightarrow e_{ss} = \frac{A}{\lim_{s \rightarrow 0} s^2G(s)H(s)} = \frac{A}{K_A}$$

Where,

$$K_A = \lim_{s \rightarrow 0} s^2G(s)H(s)$$

Types of input and steady-state error are summarized as follows.

Error Constant	Equation	Steady-state error (e_{ss})
Position Error Constant (K_p)	$K_p = \lim_{s \rightarrow 0} G(s)H(s)$	$e_{ss} = \frac{A}{1 + K_p}$
Velocity Error Constant (K_v)	$K_v = \lim_{s \rightarrow 0} sG(s)H(s)$	$e_{ss} = \frac{A}{K_v}$
Acceleration Error Constant (K_a)	$K_a = \lim_{s \rightarrow 0} s^2G(s)H(s)$	$e_{ss} = \frac{A}{K_a}$

STATIC ERROR COEFFICIENT METHOD

The general form of $G(s)H(s)$ is

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\dots(1+T_ns)}{s^j(1+T_as)(1+T_bs)\dots(1+T_ms)}$$

Here, j = no. of poles at origin ($s = 0$)

Type 0

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\dots(1+T_ns)}{(1+T_as)(1+T_bs)\dots(1+T_ms)}$$

Here,

$$K_p = \lim_{s \rightarrow 0} G(s)H(s) = K$$

Therefore,

$$e_{ss} = \frac{A}{1 + K}$$

Type 1

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\dots(1+T_ns)}{s(1+T_as)(1+T_bs)\dots(1+T_ms)}$$

Here,

$$K_v = \lim_{s \rightarrow 0} sG(s)H(s) = K$$

Therefore,

$$e_{ss} = \frac{A}{K}$$

Type 2

$$G(s)H(s) = \frac{K(1+T_1s)(1+T_2s)\dots(1+T_ns)}{s^2(1+T_as)(1+T_bs)\dots(1+T_ms)}$$

Here,

$$K_A = \lim_{s \rightarrow 0} s^2 G(s)H(s) = K$$

Therefore,

$$e_{ss} = \frac{A}{K}$$

Steady-state error and error constant for different types of input are summarized as follows.

Type	Step input		Ramp input		Parabolic input	
	K_p	e_{ss}	K_v	e_{ss}	K_A	e_{ss}
Type 0	K	$\frac{A}{1+K}$	0	∞	0	∞
Type 1	∞	0	K	$\frac{A}{K}$	0	∞
Type 2	∞	0	∞	0	K	$\frac{A}{K}$

EFFECT OF POLE AND ZERO TO TRANSFER FUNCTION

(i) Addition of a pole to the Forward Path Transfer Function:-

- Increases the order of the system
- Increases the overshoot
- Reduces stability
- Increase rise time
- Reduces bandwidth

(ii) Addition of a pole to the Closed-Loop Transfer function:-

- Increases rise time
- Decreases overshoot

(iii) Addition of a zero to the Closed-Loop Transfer function:-

- Decreases rise time
- Increases overshoot

(iv) Addition of a zero to the Forward path Transfer function:-

- Added zero far away from imaginary axis – Overshoot large & damping is very poor
- When zero moves to the right – Overshoot reduce & damping improves
- When zero moves closer to the origin – Overshoot increases & damping improves

PROPORTIONAL CONTROLLER

The proportional controller produces an output, which is proportional to error signal.

$$u(t) \propto e(t)$$

$$\Rightarrow u(t) = K_P e(t)$$

Apply Laplace transform on both the sides -

$$U(s) = K_P E(s)$$

$$\frac{U(s)}{E(s)} = K_P$$

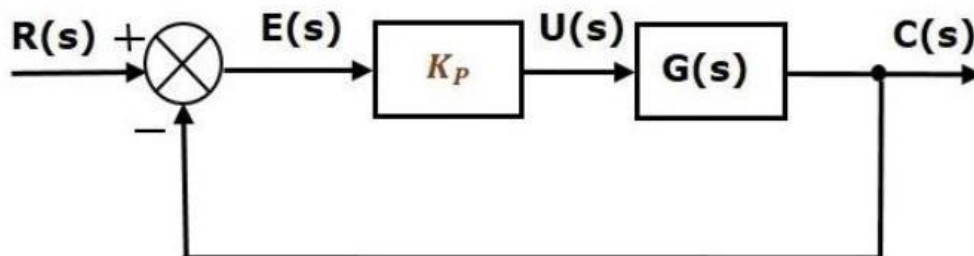
Therefore, the transfer function of the proportional controller is K_P .
Where,

$U(s)$ is the Laplace transform of the actuating signal $u(t)$

$E(s)$ is the Laplace transform of the error signal $e(t)$

K_P is the proportionality constant

The block diagram of the unity negative feedback closed loop control system along with the proportional controller is shown in the following figure.



DERIVATIVE CONTROLLER

The derivative controller produces an output, which is derivative of the error signal.

$$u(t) = K_D \frac{de(t)}{dt}$$

Apply Laplace transform on both sides.

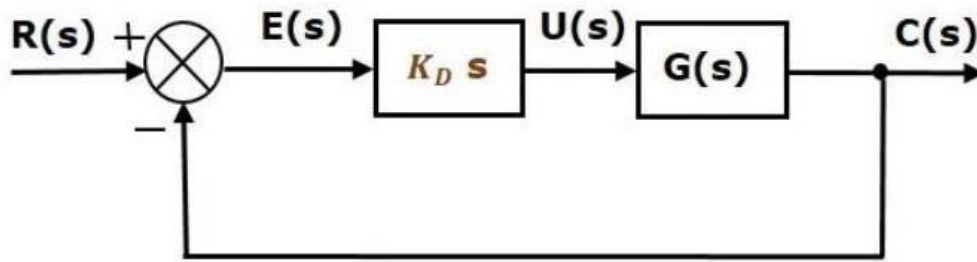
$$U(s) = K_D s E(s)$$

$$\frac{U(s)}{E(s)} = K_D s$$

Therefore, the transfer function of the derivative controller is $K_D s$

Where, K_D is the derivative constant.

The block diagram of the unity negative feedback closed loop control system along with the derivative controller is shown in the following figure.



INTEGRAL CONTROLLER

The integral controller produces an output, which is integral of the error signal.

$$u(t) = K_I \int e(t) dt$$

Apply Laplace transform on both the sides -

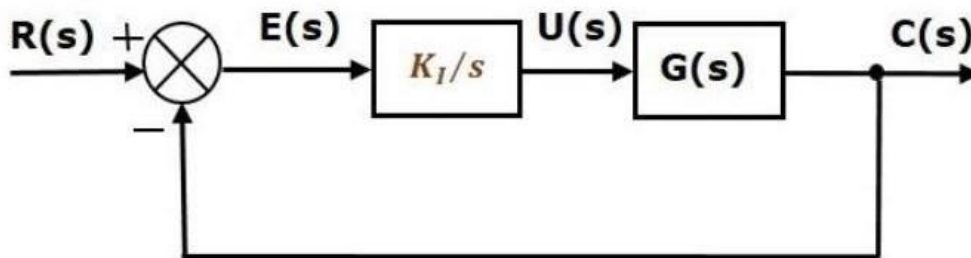
$$U(s) = \frac{K_I E(s)}{s}$$

$$\frac{U(s)}{E(s)} = \frac{K_I}{s}$$

Therefore, the transfer function of the integral controller is $\frac{K_I}{s}$.

Where, K_I is the integral constant.

The block diagram of the unity negative feedback closed loop control system along with the integral controller is shown in the following figure.



The integral controller is used to decrease the steady state error.

PROPORTIONAL DERIVATIVE (PD) CONTROLLER

The proportional derivative controller produces an output, which is the combination of the outputs of proportional and derivative controllers.

$$u(t) = K_P e(t) + K_D \frac{de(t)}{dt}$$

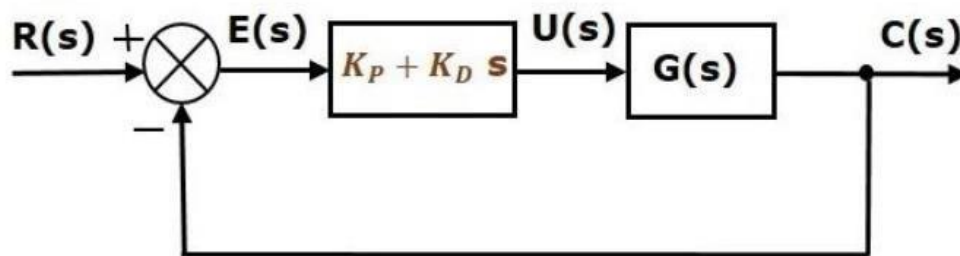
Apply Laplace transform on both sides -

$$U(s) = (K_P + K_D s) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + K_D s$$

Therefore, the transfer function of the proportional derivative controller is $K_P + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional derivative controller is shown in the following figure.



The proportional derivative controller is used to improve the stability of control system without affecting the steady state error.

PROPORTIONAL INTEGRAL (PI) CONTROLLER

The proportional integral controller produces an output, which is the combination of outputs of the proportional and integral controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt$$

Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

Therefore, the transfer function of proportional integral controller is $K_P + \frac{K_I}{s}$.

The proportional integral controller is used to decrease the steady state error without affecting the stability of the control system.

PROPORTIONAL INTEGRAL DERIVATIVE (PID) CONTROLLER

The proportional integral derivative controller produces an output, which is the combination of the outputs of proportional, integral and derivative controllers.

$$u(t) = K_P e(t) + K_I \int e(t) dt + K_D \frac{de(t)}{dt}$$

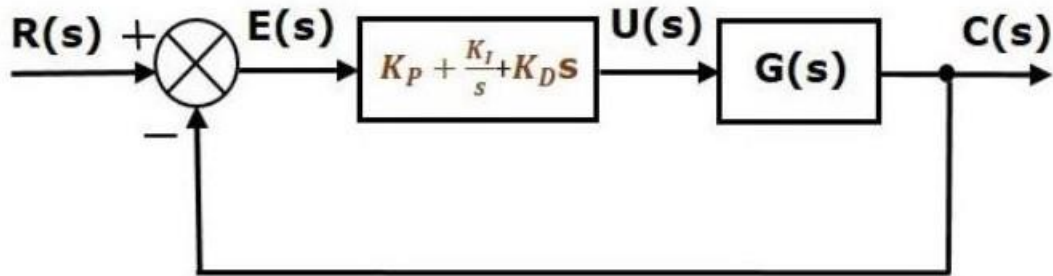
Apply Laplace transform on both sides -

$$U(s) = \left(K_P + \frac{K_I}{s} + K_D s \right) E(s)$$

$$\frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s$$

Therefore, the transfer function of the proportional integral derivative controller is $K_P + \frac{K_I}{s} + K_D s$.

The block diagram of the unity negative feedback closed loop control system along with the proportional integral derivative controller is shown in the following figure.

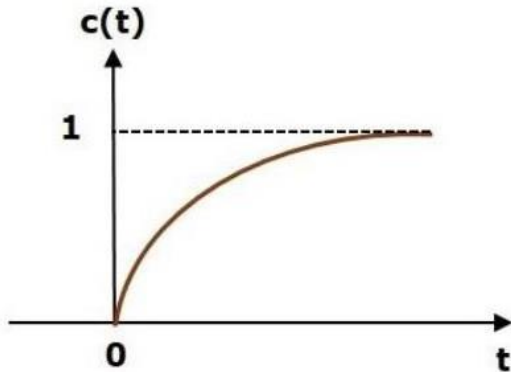


CHAPTER#3 ANALYSIS OF STABILITY BY ROOT LOCUS TECHNIQUE

STABILITY

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A **stable system** produces a bounded output for a given bounded input.

The following figure shows the response of a stable system.



This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output. We know that the unit step signal has the value of one for all positive values of t including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

TYPES OF SYSTEMS BASED ON STABILITY

We can classify the systems based on stability as follows.

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

If the system is stable for all the range of system component values, then it is known as the **absolutely stable system**. The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of ' s ' plane. Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the ' s ' plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as **conditionally stable system**.

Marginally Stable System

If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as **marginally stable system**. The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis. Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

ROUTH-HURWITZ STABILITY CRITERION

Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability. If any control system doesn't satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable. So, the sufficient condition is helpful for knowing whether the control system is stable or not.

NECESSARY CONDITION FOR ROUTH-HURWITZ STABILITY

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Consider the characteristic equation of the order 'n' is -

$$a_0s^n + a_1s^{n-1} + a_2s^{n-2} + \dots + a_{n-1}s^1 + a_ns^0 = 0$$

Note that, there should not be any term missing in the n^{th} order characteristic equation. This means that the n^{th} order characteristic equation should not have any coefficient that is of zero value.

SUFFICIENT CONDITION FOR ROUTH-HURWITZ STABILITY

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

ROUTH ARRAY METHOD

If all the roots of the characteristic equation exist to the left half of the 's' plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the 's' plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.

So, to overcome this problem there we have the **Routh array method**. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the 's' plane and the control system is unstable.

Follow this procedure for forming the Routh table.

- Fill the first two rows of the Routh array with the coefficients of the characteristic polynomial as mentioned in the table below. Start with the coefficient of s^n and continue up to the coefficient of s_0 .
- Fill the remaining rows of the Routh array with the elements as mentioned in the table below. Continue this process till you get the first column element of **row** s_0 .

Note – If any row elements of the Routh table have some common factor, then you can divide the row elements with that factor for the simplification will be easy.

The following table shows the Routh array of the n^{th} order characteristic polynomial.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6
s^{n-1}	a_1	a_3	a_5	a_7
s^{n-2}	b_1 $= \frac{a_1 a_2 - a_3 a_0}{a_1}$	b_2 $= \frac{a_1 a_4 - a_5 a_0}{a_1}$	b_3 $= \frac{a_1 a_6 - a_7 a_0}{a_1}$
s^{n-3}	c_1 $= \frac{b_1 a_3 - b_2 a_1}{b_1}$	c_2 $= \frac{b_1 a_5 - b_2 a_1}{b_1}$	\vdots			
\vdots	\vdots	\vdots	\vdots			
s^1	\vdots	\vdots				
s^0	a_n					

Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Step 1 – Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial,

$$s^4 + 3s^3 + 3s^2 + 2s + 1$$

are positive. So, the control system satisfies the necessary condition.

Step 2 – Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3 \times 3) - (2 \times 1)}{3} = \frac{7}{3}$	$\frac{(3 \times 1) - (0 \times 1)}{3} = \frac{3}{3} = 1$	
s^1	$\frac{(\frac{7}{3} \times 2) - (1 \times 3)}{\frac{7}{3}} = \frac{5}{7}$		
s^0	1		

Step 3 – Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in

the first column of the Routh array. So, the control system is stable.

SPECIAL CASES OF ROUTH ARRAY

We may come across two types of situations, while forming the Routh table. It is difficult to complete the Routh table from these two situations.

The two special cases are –

- The first element of any row of the Routh's array is zero.
- All the elements of any row of the Routh's array are zero.

Let us now discuss how to overcome the difficulty in these two cases, one by one.

➤ **First Element of any row of the Routh's array is zero**

If any row of the Routh's array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ . And then continue the process of completing the Routh's table. Now, find the number of sign changes in the first column of the Routh's table by substituting ϵ tends to zero.

➤ **All the Elements of any row of the Routh's array are zero**

In this case, follow these two steps –

- Write the auxiliary equation, $A(s)$ of the row, which is just above the row of zeros.
- Differentiate the auxiliary equation, $A(s)$ with respect to s . fill the row of zeros with these coefficients.

ROOT LOCUS

The Root locus is the locus of the roots of the characteristic equation by varying system gain K from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1 + G(s)H(s) = 0$$

We can represent $G(s)H(s)$ as

$$G(s)H(s) = K \frac{N(s)}{D(s)}$$

Where,

- K represents the multiplying factor

- $N(s)$ represents the numerator term having (factored) n^{th} order polynomial of 's'.
- $D(s)$ represents the denominator term having (factored) m^{th} order polynomial of 's'.

Substitute, $G(s)H(s)$ value in the characteristic equation.

$$1 + k \frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

Case 1 – $K = 0$

If $K = 0$, then $D(s) = 0$.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 – $K = \infty$

Re-write the above characteristic equation as

$$K \left(\frac{1}{K} + \frac{N(s)}{D(s)} \right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Substitute, $K = \infty$ in the above equation.

$$\frac{1}{\infty} + \frac{N(s)}{D(s)} = 0 \Rightarrow \frac{N(s)}{D(s)} = 0 \Rightarrow N(s) = 0$$

If $K = \infty$, then $N(s) = 0$. It means the closed loop poles are equal to the open loop zeros when K is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

ANGLE CONDITION AND MAGNITUDE CONDITION

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of $G(s)H(s)$ is

$$\angle G(s)H(s) = \tan^{-1} \left(\frac{0}{-1} \right) = (2n + 1)\pi$$

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180° .

Magnitude of $G(s)H(s)$ is –

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

THE ROOT LOCUS IS A GRAPHICAL REPRESENTATION IN S-DOMAIN AND IT IS SYMMETRICAL ABOUT THE REAL AXIS. Because the open loop poles and zeros exist in the s-domain having the values either as real or as complex conjugate pairs.

RULES FOR CONSTRUCTION OF ROOT LOCUS

Follow these rules for constructing a root locus.

Rule 1 – Locate the open loop poles and zeros in the 's' plane.

Rule 2 – Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N=P \text{ if } P \geq Z$$

$$N=Z \text{ if } P < Z$$

Rule 3 – Identify and draw the **real axis root locus branches**.

If the angle of the open loop transfer function at a point is an odd multiple of 180° , then that point is on the root locus. If odd number of the open loop poles and zeros exist to the left side of a point on the real axis, then that point is on the root locus branch. Therefore, the branch of points which satisfies this condition is the real axis of the root locus branch.

Rule 4 – Find the centroid and the angle of asymptotes.

- If $P=Z$, then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If $P>Z$, then Z number of root locus branches start at finite open loop poles and end at finite open loop zeros and $P-Z$ number of root locus branches start at finite open loop poles and end at infinite open loop zeros.
- If $P<Z$, then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and $Z-P$ number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

So, some of the root locus branches approach infinity, when $P \neq Z$. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid α** by using this formula,

$$\alpha = \frac{\sum \text{Real part of finite open loop poles} - \sum \text{Real part of finite open loop zeros}}{P - Z}$$

The formula for the angle of **asymptotes** θ is

$$\theta = \frac{(2q + 1)180^\circ}{P - Z}$$

Where,

$$q = 0, 1, 2, \dots, (P - Z) - 1$$

Rule 5 – Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rule 6 – Find Break-away and Break-in points.

- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note – Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write **K** in terms of **s** from the characteristic equation $1 + G(s)H(s) = 0$.
- Differentiate **K** with respect to **s** and make it equal to zero. Substitute these values of **s** in the above equation.
- The values of **s** for which the **K** value is positive are the **break points**.

Rule 7 – Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕ_d is

$$\phi_d = 180^\circ - \phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a = 180^\circ + \phi$$

Where,

$$\phi = \sum \phi_P - \sum \phi_Z$$

Example

Let us now draw the root locus of the control system having open loop transfer function,

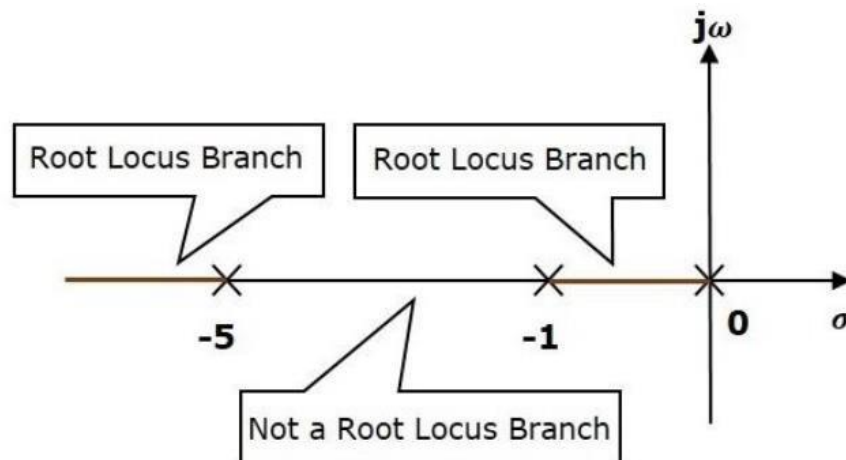
$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

Step 1 – The given open loop transfer function has three poles at $s = 0$,

$s = -1$, $s = -5$. It doesn't have any zero. Therefore, the number of root locus branches is equal

to the number of poles of the open loop transfer function.

$$N=P=3$$

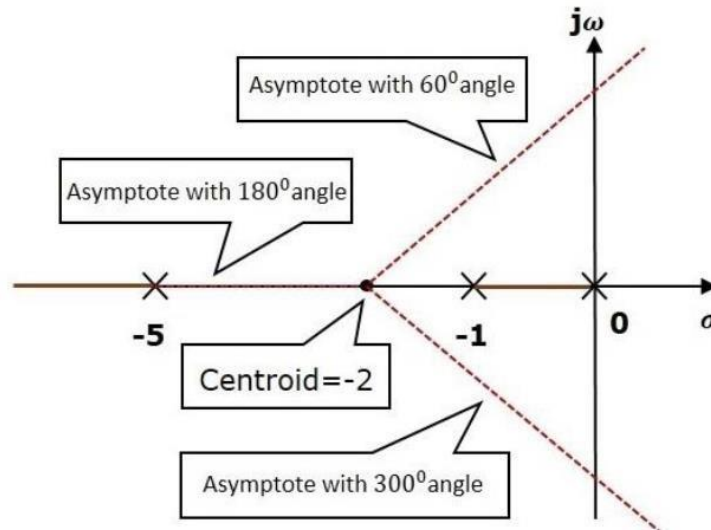


The three poles are located as shown in the above figure. The line segment between $s=-1$, and $s=0$ is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of $s=-5$.

Step 2 – We will get the values of the centroid and the angle of asymptotes by using the given formulae.

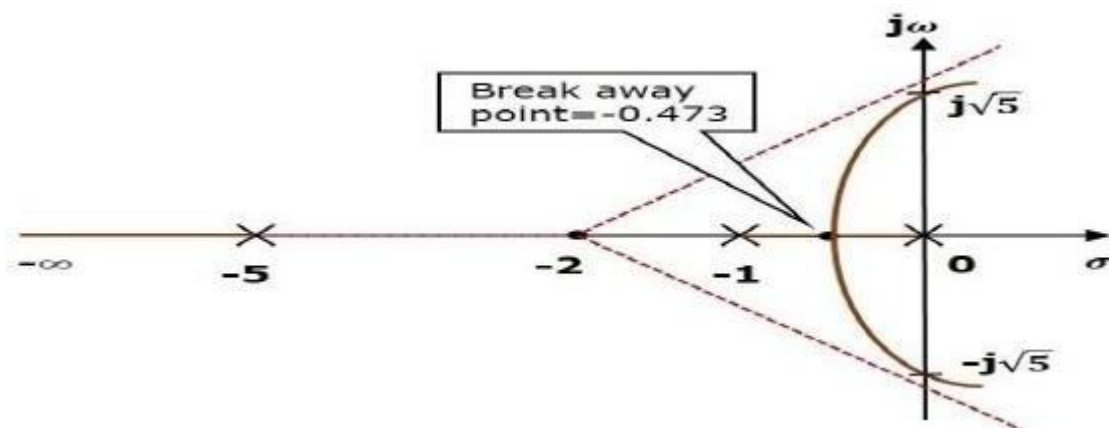
$$\text{Centroid} = \frac{0 - 5 - 1}{3} = -2$$

The angle of asymptotes are $\theta = 60^\circ, 180^\circ$ and 300° .



Step 3 – Since two asymptotes have the angles of 60 and 300, two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii), the root locus branches intersects the imaginary axis at $j\sqrt{5}$ and $-j\sqrt{5}$.

There will be one break-away point on the real axis root locus branch between the poles $s = -1$ and $s = -5$. By following the procedure given for the calculation of break-away point, we will get it as $s = -0.473$.



EFFECTS OF ADDING OPEN LOOP POLES AND ZEROS ON ROOT LOCUS

The root locus can be shifted in 's' plane by adding the open loop poles and the open loop zeros.

- If we include a pole in the open loop transfer function, then some of root locus branches will move towards right half of 's' plane. Because of this, the damping ratio δ decreases. Which implies, damped frequency ω_d increases and the time domain specifications like delay time t_d , rise time t_r and peak time t_p decrease. But, it effects the system stability.
- If we include a zero in the open loop transfer function, then some of root locus branches will move towards left half of 's' plane. So, it will increase the control system stability. In this case, the damping ratio δ increases. Which implies, damped frequency ω_d decreases and the time domain specifications like delay time t_d , rise time t_r and peak time t_p increase.

CHAPTER#4 FREQUENCY RESPONSE ANALYSIS

FREQUENCY RESPONSE

The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steady state response of a system for an input sinusoidal signal is known as the **frequency response**. In this chapter, we will focus only on the steady state response.

If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles. Let the input signal be

$$r(t) = A \sin(\omega_0 t)$$

The open loop transfer function will be –

$$G(s) = G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

Substitute, $\omega = \omega_0$ in the above equation.

$$G(j\omega_0) = |G(j\omega_0)| \angle G(j\omega_0)$$

The output signal is

$$c(t) = A |G(j\omega_0)| \sin(\omega_0 t + \angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega = \omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega = \omega_0$.

Where,

A is the amplitude of the input sinusoidal signal.

ω_0 is angular frequency of the input sinusoidal signal.

We can write, angular frequency ω_0 as shown below.

$$\omega_0 = 2\pi f_0$$

Here, f_0 is the frequency of the input sinusoidal signal. Similarly, you can follow the same procedure for closed loop control system.

FREQUENCY DOMAIN SPECIFICATIONS

The frequency domain specifications are

- ❖ Resonant peak
- ❖ Resonant frequency
- ❖ Bandwidth.

Consider the transfer function of the second order closed control system as

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Substitute, $s = j\omega$ in the above equation.

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow T(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega\omega_n + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Let, $\frac{\omega}{\omega_n} = u$ Substitute this value in the above equation.

$$T(j\omega) = \frac{1}{(1 - u^2) + j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1 - u^2)^2 + (2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega) = -\tan^{-1} \left(\frac{2\delta u}{1 - u^2} \right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r . At $\omega = \omega_r$, the first derivate of the magnitude of $T(j\omega)$ is zero.

Differentiate M with respect to u .

$$\begin{aligned}\frac{dM}{du} &= -\frac{1}{2} [(1 - u^2)^2 + (2\delta u)^2]^{-\frac{3}{2}} [2(1 - u^2)(-2u) + 2(2\delta u)(2\delta)] \\ \Rightarrow \frac{dM}{du} &= -\frac{1}{2} [(1 - u^2)^2 + (2\delta u)^2]^{-\frac{3}{2}} [4u(u^2 - 1 + 2\delta^2)]\end{aligned}$$

Substitute, $u = u_r$ and $\frac{dM}{du} = 0$ in the above equation.

$$\begin{aligned}0 &= -\frac{1}{2} [(1 - u_r^2)^2 + (2\delta u_r)^2]^{-\frac{3}{2}} [4u_r(u_r^2 - 1 + 2\delta^2)] \\ &\Rightarrow 4u_r(u_r^2 - 1 + 2\delta^2) = 0 \\ &\Rightarrow u_r^2 - 1 + 2\delta^2 = 0 \\ &\Rightarrow u_r^2 = 1 - 2\delta^2 \\ &\Rightarrow u_r = \sqrt{1 - 2\delta^2}\end{aligned}$$

Substitute, $u_r = \frac{\omega_r}{\omega_n}$ in the above equation.

$$\begin{aligned}\frac{\omega_r}{\omega_n} &= \sqrt{1 - 2\delta^2} \\ \Rightarrow \omega_r &= \omega_n \sqrt{1 - 2\delta^2}\end{aligned}$$

Resonant Peak:

It is the peak (maximum) value of the magnitude of $T(j\omega)$. It is denoted by M_r . At $u = u_r$, the Magnitude of $T(j\omega)$ is -

$$M_r = \frac{1}{\sqrt{(1 - u_r^2)^2 + (2\delta u_r)^2}}$$

Substitute, $u_r = \sqrt{1 - 2\delta^2}$ and $1 - u_r^2 = 2\delta^2$ in the above equation.

$$\begin{aligned}M_r &= \frac{1}{\sqrt{(2\delta^2)^2 + (2\delta\sqrt{1 - 2\delta^2})^2}} \\ \Rightarrow M_r &= \frac{1}{2\delta\sqrt{1 - \delta^2}}\end{aligned}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio δ . So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth:

It is the range of frequencies over which, the magnitude of $T(j\omega)$ drops to 70.7% from its zero frequency value.

At $\omega=0$, the value of u will be zero.

Substitute, $u=0$ in M .

$$M = \frac{1}{\sqrt{(1 - 0^2)^2 + (2\delta(0))^2}} = 1$$

Therefore, the magnitude of $T(j\omega)$ is one at $\omega=0$

At 3-dB frequency, the magnitude of $T(j\omega)$ will be 70.7% of magnitude of $T(j\omega)$ at $\omega=0$
i.e., at $\omega = \omega_B$, $M = 0.707(1) = \frac{1}{\sqrt{2}}$

$$\Rightarrow M = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{(1 - u_b^2)^2 + (2\delta u_b)^2}}$$

$$\Rightarrow 2 = (1 - u_b^2)^2 + (2\delta)^2 u_b^2$$

Let, $u_b^2 = x$

$$\Rightarrow 2 = (1 - x)^2 + (2\delta)^2 x$$

$$\Rightarrow x^2 + (4\delta^2 - 2)x - 1 = 0$$

$$\Rightarrow x = \frac{-(4\delta^2 - 2) \pm \sqrt{(4\delta^2 - 2)^2 + 4}}{2}$$

Consider only the positive value of x .

$$x = 1 - 2\delta^2 + \sqrt{(2\delta^2 - 1)^2 + 1}$$

$$\Rightarrow x = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

Substitute, $x = u_b^2 = \frac{\omega_b^2}{\omega_n^2}$

$$\frac{\omega_b^2}{\omega_n^2} = 1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}$$

$$\Rightarrow \omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}}$$

Bandwidth ω_b in the frequency response is inversely proportional to the rise time t_r in the time domain transient response.

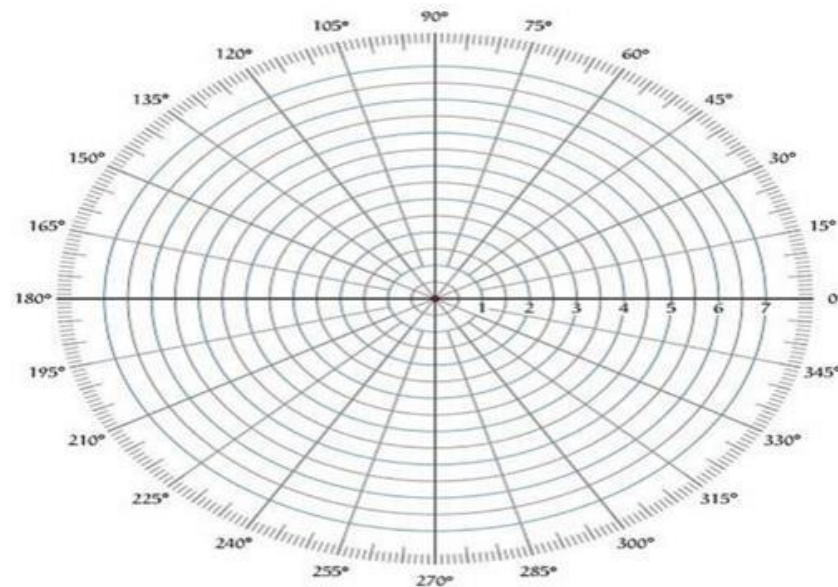
POLAR PLOTS

Polar plot is a plot which can be drawn between magnitude and phase. Here, the magnitudes are represented by normal values only.

The polar form of $G(j\omega)H(j\omega)$ is

$$G(j\omega)H(j\omega) = |G(j\omega)H(j\omega)| \angle G(j\omega)H(j\omega)$$

The **Polar plot** is a plot, which can be drawn between the magnitude and the phase angle of $G(j\omega)H(j\omega)$ by varying ω from zero to ∞ . The polar graph sheet is shown in the following figure.



This graph sheet consists of concentric circles and radial lines. The **concentric circles** and the **radial lines** represent the magnitudes and phase angles respectively. These angles are represented by positive values in anti-clock wise direction. Similarly, we can represent angles with negative values in clockwise direction. For example, the angle 270^0 in anti-clock wise direction is equal to the angle -90^0 in clockwise direction.

RULES FOR DRAWING POLAR PLOTS

Follow these rules for plotting the polar plots.

- Substitute, $s=j\omega$ in the open loop transfer function.
- Write the expressions for magnitude and the phase of $G(j\omega)H(j\omega)$
- Find the starting magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting $\omega=0$. So, the polar plot starts with this magnitude and the phase angle.
- Find the ending magnitude and the phase of $G(j\omega)H(j\omega)$ by substituting $\omega=\infty$ So, the polar plot ends with this magnitude and the phase angle.
- Check whether the polar plot intersects the real axis, by making the imaginary term of $G(j\omega)H(j\omega)$ equal to zero and find the value(s) of ω .
- Check whether the polar plot intersects the imaginary axis, by making real term of $G(j\omega)H(j\omega)$ equal to zero and find the value(s) of ω .
- For drawing polar plot more clearly, find the magnitude and phase of $G(j\omega)H(j\omega)$ by considering the other value(s) of ω .

Example:

Consider the open loop transfer function of a closed loop control system.

$$G(s)H(s) = \frac{5}{s(s+1)(s+2)}$$

Let us draw the polar plot for this control system using the above rules.

Step 1 – Substitute, $s = j\omega$ in the open loop transfer function.

$$G(j\omega)H(j\omega) = \frac{5}{j\omega(j\omega+1)(j\omega+2)}$$

The magnitude of the open loop transfer function is

$$M = \frac{5}{\omega(\sqrt{\omega^2+1})(\sqrt{\omega^2+4})}$$

The phase angle of the open loop transfer function is

$$\phi = -90^\circ - \tan^{-1} \omega - \tan^{-1} \frac{\omega}{2}$$

Frequency (rad/sec)	Magnitude	Phase angle(degrees)
0	∞	-90 or 270
∞	0	-270 or 90

So, the polar plot starts at $(\infty, -90^\circ)$ and ends at $(0, -270^\circ)$. The first and the second terms within the brackets indicate the magnitude and phase angle respectively.

Step 3 – Based on the starting and the ending polar co-ordinates, this polar plot will intersect the negative real axis. The phase angle corresponding to the negative real axis is -180° or 180° . So, by equating the phase angle of the open loop transfer function to either -180° or 180° , we will get the ω value as $\sqrt{2}$.

By substituting $\omega = \sqrt{2}$ in the magnitude of the open loop transfer function, we will get $M = 0.83$. Therefore, the polar plot intersects the negative real axis when $\omega = \sqrt{2}$ and the polar coordinate is $(0.83, -180^\circ)$.

So, we can draw the polar plot with the above information.

BODE PLOTS

The Bode plot or the Bode diagram consists of two plots –

- Magnitude plot
- Phase plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, y-axis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB is -

$$M = 20 \log |G(j\omega)H(j\omega)|$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi = \angle G(j\omega)H(j\omega)$$

Basic of Bode Plots:

The following table shows the slope, magnitude and the phase angle values of the terms present in the open loop transfer function. This data is useful while drawing the Bode plots.

Type of term	$G(j\omega)H(j\omega)$	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20 \log K$	0
Zero at origin	$j\omega$	20	$20 \log \omega$	90
'n' zeros at origin	$(j\omega)^n$	$20 n$	$20 n \log \omega$	$90 n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20 \log \omega$	-90 or 270
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20 n$	$-20 n \log \omega$	$-90 n$ or $270 n$
Simple zero	$1 + j\omega r$	20	0 for $\omega < \frac{1}{r}$ $20 \log \omega r$ for $\omega > \frac{1}{r}$	0 for $\omega < \frac{1}{r}$ 90 for $\omega > \frac{1}{r}$
Simple pole	$\frac{1}{1+j\omega r}$	-20	0 for $\omega < \frac{1}{r}$ $-20 \log \omega r$ for $\omega > \frac{1}{r}$	0 for $\omega < \frac{1}{r}$ -90 or 270 for $\omega > \frac{1}{r}$
Second order derivative term	$\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\omega}{\omega_n} \right)$	40	$40 \log \omega_n$ for $\omega < \omega_n$ $20 \log (2\delta\omega_n^2)$ for $\omega = \omega_n$ $40 \log \omega$ for $\omega > \omega_n$	0 for $\omega < \omega_n$ 90 for $\omega = \omega_n$ 180 for $\omega > \omega_n$
Second order integral term	$\frac{1}{\omega_n^2 \left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\omega}{\omega_n} \right)}$	-40	$-40 \log \omega_n$ for $\omega < \omega_n$ $-20 \log (2\delta\omega_n^2)$ for $\omega = \omega_n$ $-40 \log \omega$ for $\omega > \omega_n$	-0 for $\omega < \omega_n$ -90 for $\omega = \omega_n$ -180 for $\omega > \omega_n$

Case-1:

Consider the open loop transfer function $G(s)H(s) = K$.

Magnitude $M = 20 \log K$ dB

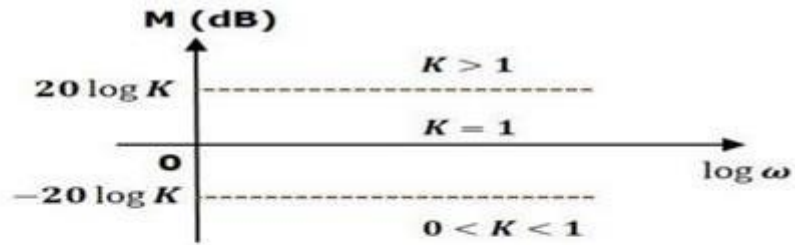
Phase angle $\phi = 0$ degrees

If $K = 1$, then magnitude is 0 dB.

If $K > 1$, then magnitude will be positive.

If $K < 1$, then magnitude will be negative.

The following figure shows the corresponding Bode plot.



The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself - is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift $20 \log K$ dB above the 0 dB line. For the negative values of K, the horizontal line will shift $20 \log K$ dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.

Case-2:

Consider the open loop transfer function $G(s)H(s) = S$

Magnitude $M = 20 \log \omega$ dB

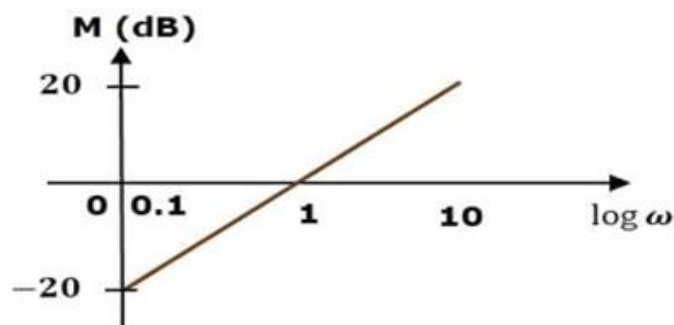
Phase angle $\phi = 90^\circ$

At $\omega = 0.1$ rad/sec, the magnitude is -20 dB.

At $\omega = 1$ rad/sec, the magnitude is 0 dB.

At $\omega = 10$ rad/sec, the magnitude is 20 dB.

The following figure shows the corresponding Bode plot.



The magnitude plot is a line, which is having a slope of 20 dB/dec. This line started at $\omega = 0.1$ rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at $\omega = 1$ rad/sec. In this case, the phase plot is 90° line.

Case-3:

Consider the open loop transfer function $G(s) H(s)=1+s\tau$.

$$\text{Magnitude} = \sqrt{1+(s\tau)^2}$$

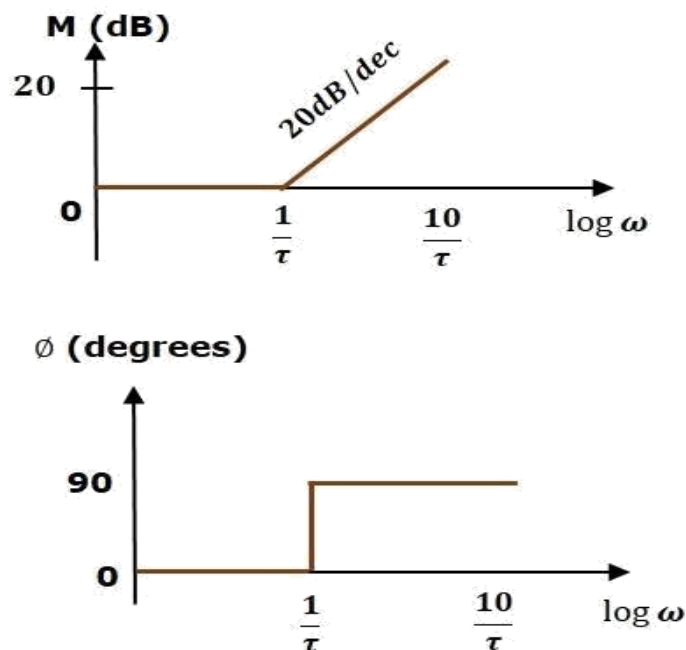
Phase angle =

$$\phi = \tan^{-1} \omega\tau \text{ degrees}$$

For $\omega < \frac{1}{\tau}$, the magnitude is 0 dB and phase angle is 0 degrees.

For $\omega > \frac{1}{\tau}$, the magnitude is $20\log\omega\tau$ dB and phase angle is 90° .

The following figure shows the corresponding Bode plot



The magnitude plot is having magnitude of 0 dB up to $\omega=1\tau$ rad/sec. From $\omega=1\tau$ rad/sec, it is having a slope of 20 dB/decade. In this case, the phase plot is having phase angle of 0 degrees up to $\omega=1\tau$ rad/sec and from here, it is having phase angle of 90° . This Bode plot is called the **asymptotic Bode plot**. As the magnitude and the phase plots are represented with straight lines, the Exact Bode plots resemble the asymptotic Bode plots. The only difference is that the Exact Bode plots will have simple curves instead of straight lines.

RULES FOR CONSTRUCTION OF BODE PLOTS:

Follow these rules while constructing a Bode plot.

- Represent the open loop transfer function in the standard time constant form.
- Substitute, $s=j\omega$ in the above equation.
- Find the corner frequencies and arrange them in ascending order.
- Consider the starting frequency of the Bode plot as $1/10^{\text{th}}$ of the minimum corner frequency or 0.1 rad/sec whichever is smaller value and draw the Bode plot upto 10 times maximum corner frequency.
- Draw the magnitude plots for each term and combine these plots properly.
- Draw the phase plots for each term and combine these plots properly.

STABILITY ANALYSIS USING BODE PLOTS

From the Bode plots, we can say whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency:

The frequency at which the phase plot is having the phase of -180° is known as **phase cross over frequency**. It is denoted by ω_{pc} . The unit of phase cross over frequency is **rad/sec**.

Gain Cross over Frequency:

The frequency at which the magnitude plot is having the magnitude of zero dB is known as **gain cross over frequency**. It is denoted by ω_{gc} . The unit of gain cross over frequency is **rad/sec**.

The stability of the control system based on the relation between the phase cross over frequency and the gain cross over frequency is listed below.

- If the phase cross over frequency ω_{pc} is greater than the gain cross over frequency ω_{gc} , then the control system is **stable**.
- If the phase cross over frequency ω_{pc} is equal to the gain cross over frequency ω_{gc} , then the control system is **marginally stable**.
- If the phase cross over frequency ω_{pc} is less than the gain cross over frequency ω_{gc} , then the control system is **unstable**.

Gain Margin:

Gain margin GM is equal to negative of the magnitude in dB at phase cross over frequency.

$$GM = -20\log(M_{pc})$$

Where, M_{pc} is the magnitude at phase cross over frequency. The unit of gain margin (GM) is **dB**.

Phase Margin:

The formula for phase margin PM is $PM = 180^\circ + \phi_{gc}$

Where, ϕ_{gc} is the phase angle at gain cross over frequency. The unit of phase margin is **degrees**.

********The stability of the control system based on the relation between gain margin and phase margin is listed below.

- If both the gain margin GM and the phase margin PM are positive, then the control system is **stable**.
- If both the gain margin GM and the phase margin PM are equal to zero, then the control system is **marginally stable**.
- If the gain margin GM and / or the phase margin PM are/is negative, then the control system is **unstable**.

Example: Sketch the Bode plot for the Transfer function

$$G(s) = \frac{1000}{(1+0.1s)(1+0.001s)}$$

Determine the a) Phase Margin

- b) Gain Margin
- c) Stability of the System

Solution: Step-1 Put $s = j\omega$

$$G(j\omega) = \frac{1000}{(1+j0.1\omega)(1+j0.001\omega)}$$

The given transfer function is of type '0' system. Therefore the initial slope of the Bode plots 0 db/decade. The starting point is given by.

$$20 \log_{10} K = 20 \log_{10} 1000 = 60 \text{ db}$$

$$\text{Corner frequencies } \omega_1 = \frac{1}{0.1} = 10 \text{ rad/sec.}$$

$$\omega_2 = \frac{1}{0.001} = 1000 \text{ rad/sec.}$$

Step 2 : Mark the starting point 60 db on y-axis and draw a line of slope 0 db/decade up to first corner frequency.

Step 3 : From first corner frequency to second corner frequency draw a line with slope $(0 - 20) = -20$ db/decade).

Step 4 : From second corner frequency to next corner frequency (if given) draw a line having the slope $-20 + (-20) = -40$ db/decade.

Step 5 : The magnitude plot is complete and now draw the phase plot by calculating the phase at different frequencies (as given in table).

Step 6: From the bode plot

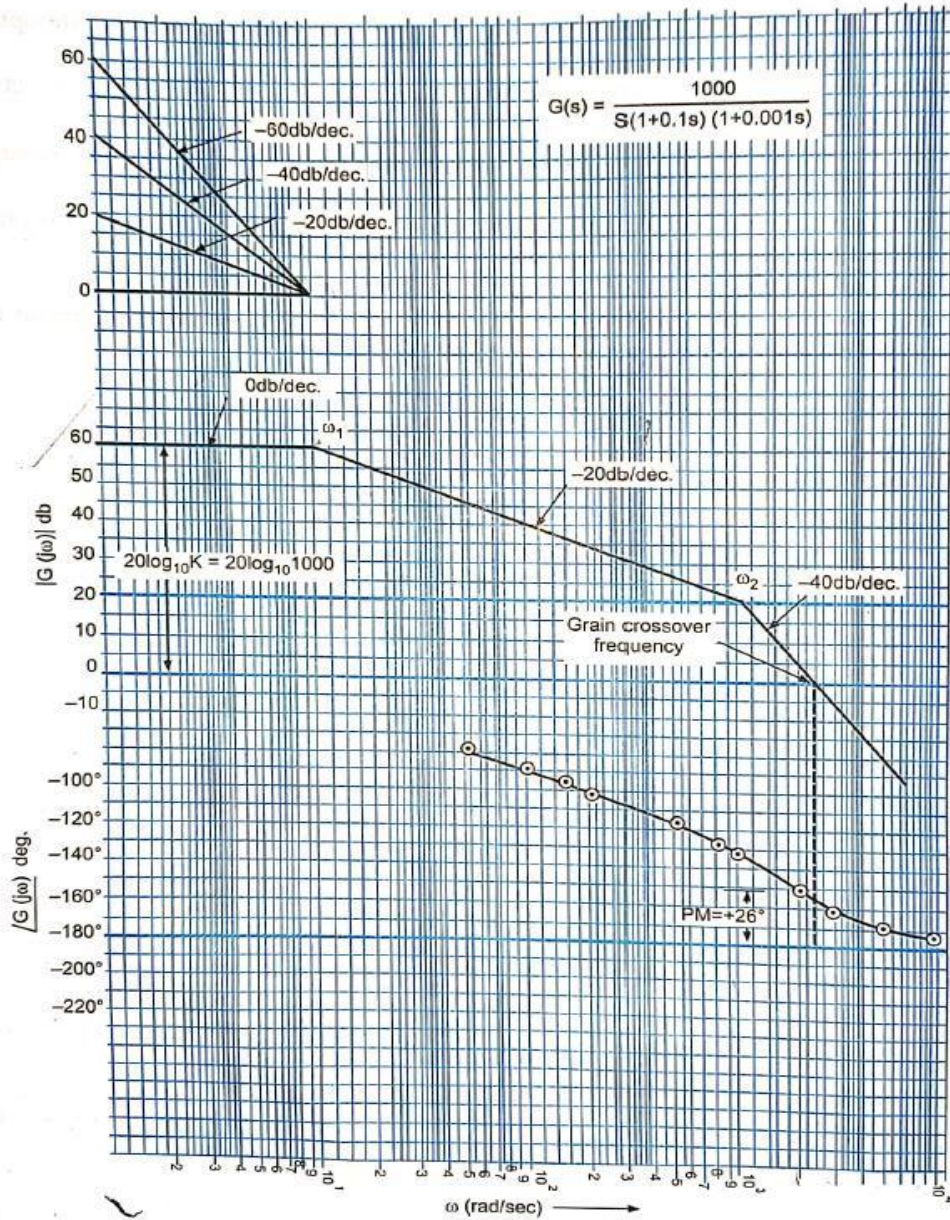
From the point of intersection of magnitude curve with 0 db axis draw a line on phase curve. This line cuts the phase curve at -154°

$$\begin{aligned} \text{P.M} &= -154 - (-180) \\ &= +26^\circ \end{aligned}$$

Step 7: Gain margin $G.M = \infty$

Since, $\text{P.M} = +26^\circ$ and gain margin $= \infty$, the system is inherently stable.

ω	$-\text{Arg}(1 + j0.1\omega)$ $-\tan^{-1}(0.1\omega)$	$-\text{Arg}(1 + j0.001\omega)$ $-\tan^{-1}(0.001\omega)$	Resultant
50	-78.6°	-2.86°	-81.46°
100	-84.2°	-5.7°	-90°
150	-86.2°	-8.5°	-94°
200	-87.13°	-11.3°	-98°
500	-88.85°	-26.56°	-115.4°
800	-89.28°	-38.65°	-127.93°
1000	-89.48°	-45°	-134.42°
2000	-89.72°	-63.43°	-153.15°
3000	-89.8°	-71.56°	-161.36°
5000	-89.88°	-78.69°	-168.57°
8000	-89.92°	-82.87°	-172.79°



CLOSED LOOP FREQUENCY RESPONSE

Consider transfer function $\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$

For unity feedback $H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Put $s = j\omega$

$$\frac{C(j\omega)}{R(j\omega)} = \frac{G(j\omega)}{1 + G(j\omega)}$$

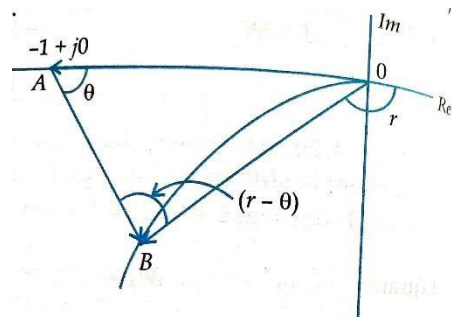
From figure

$$\vec{OB} = G(j\omega)$$

$$\vec{OA} = -1$$

$$\vec{AB} = \vec{OB} - \vec{OA} = G(j\omega) - (-1)$$

$$\vec{AB} = 1 + G(j\omega)$$



From above equation

$$\left| \frac{C(j\omega)}{R(j\omega)} \right| = M(\omega) = \frac{\vec{OB}}{\vec{AB}}$$

$$\angle \frac{C(j\omega)}{R(j\omega)} = \frac{\angle \vec{OB}}{\angle \vec{AB}} = \frac{r}{\theta} = r - \theta$$

$$\therefore \frac{C(j\omega)}{R(j\omega)} = M(\omega) e^{j\phi(\omega)}$$

where $M(j\omega)$ is the magnitude and $\phi(\omega) = r - \theta$

Frequency response consists of 2 parts: (1) magnitude (2) phase angle. Both can be plotted against different values of ω .

CHAPTER#5 NYQUIST PLOT

NYQUIST PLOT

Nyquist plots are the continuation of polar plots for finding the stability of the closed loop control systems by varying ω from $-\infty$ to ∞ . That means, Nyquist plots are used to draw the complete frequency response of the open loop transfer function.

NYQUIST STABILITY CRITERION

The Nyquist stability criterion works on the **principle of argument**. It states that if there are P poles and Z zeros are enclosed by the 's' plane closed path, then the corresponding $G(s)H(s)$ plane must encircle the origin $P-Z$ times. So, we can write the number of encirclements N as,

$$N=P-Z$$

- ❖ If the enclosed 's' plane closed path contains only poles, then the direction of the encirclement in the $G(s)H(s)$ plane will be opposite to the direction of the enclosed closed path in the 's' plane.
- ❖ If the enclosed 's' plane closed path contains only zeros, then the direction of the encirclement in the $G(s)H(s)$ plane will be in the same direction as that of the enclosed closed path in the 's' plane.

Let us now apply the principle of argument to the entire right half of the 's' plane by selecting it as a closed path. This selected path is called the **Nyquist** contour.

We know that the closed loop control system is stable if all the poles of the closed loop transfer function are in the left half of the 's' plane. So, the poles of the closed loop transfer function are nothing but the roots of the characteristic equation. As the order of the characteristic equation increases, it is difficult to find the roots. So, let us correlate these roots of the characteristic equation as follows.

- The Poles of the characteristic equation are same as that of the poles of the open loop transfer function.
- The zeros of the characteristic equation are same as that of the poles of the closed loop transfer function.

We know that the open loop control system is stable if there is no open loop pole in the right half of the 's' plane. i.e. $P=0 \Rightarrow N=-Z$

We know that the closed loop control system is stable if there is no closed loop pole in the right half of the 's' plane. i.e. $Z=0 \Rightarrow N=P$

Nyquist stability criterion states the number of encirclements about the critical point $(1+j0)$ must be equal to the poles of characteristic equation, which is nothing but the poles of the open loop transfer function in the right half of the 's' plane. The shift in origin to $(1+j0)$ gives the characteristic equation plane.

RULES FOR DRAWING NYQUIST PLOTS

Follow these rules for plotting the Nyquist plots.

- Locate the poles and zeros of open loop transfer function $G(s)H(s)$ in 's' plane.
- Draw the polar plot by varying ω from zero to infinity. If pole or zero present at $s = 0$, then varying ω from 0^+ to infinity for drawing polar plot.
- Draw the mirror image of above polar plot for values of ω ranging from $-\infty$ to zero (0^- if any pole or zero present at $s=0$).
- The number of infinite radius half circles will be equal to the number of poles or zeros at origin. The infinite radius half circle will start at the point where the mirror image of the polar plot ends. And this infinite radius half circle will end at the point where the polar plot starts.

After drawing the Nyquist plot, we can find the stability of the closed loop control system using the Nyquist stability criterion. If the critical point $(-1+j0)$ lies outside the encirclement, then the closed loop control system is absolutely stable.

STABILITY ANALYSIS USING NYQUIST PLOTS

From the Nyquist plots, we can identify whether the control system is stable, marginally stable or unstable based on the values of these parameters.

- Gain cross over frequency and phase cross over frequency
- Gain margin and phase margin

Phase Cross over Frequency

The frequency at which the Nyquist plot intersects the negative real axis (phase angle is 180°) is known as the **phase cross over frequency**. It is denoted by ω_{pc} .

Gain Cross over Frequency

The frequency at which the Nyquist plot is having the magnitude of one is known as the **gain cross over frequency**. It is denoted by ω_{gc} .

The stability of the control system based on the relation between phase cross over frequency and gain cross over frequency is listed below.

- ❖ If the phase cross over frequency ω_{pc} is greater than the gain cross over frequency ω_{gc} , then the control system is **stable**.
- ❖ If the phase cross over frequency ω_{pc} is equal to the gain cross over frequency ω_{gc} , then the control system is **marginally stable**.
- ❖ If phase cross over frequency ω_{pc} is less than gain cross over frequency ω_{gc} , then the control system is **unstable**.

Gain Margin

The gain margin GM is equal to the reciprocal of the magnitude of the Nyquist plot at the phase cross over frequency.

$$GM = \frac{1}{M_{pc}}$$

Where, M_{pc} is the magnitude in normal scale at the phase cross over frequency.

Phase Margin

The phase margin PM is equal to the sum of 180° and the phase angle at the gain cross over frequency.

$$PM = 180^\circ + \phi_{gc}$$

Where, ϕ_{gc} is the phase angle at the gain cross over frequency.

The stability of the control system based on the relation between the gain margin and the phase margin is listed below.

- ❖ If the gain margin GM is greater than one and the phase margin PM is positive, then the control system is **stable**.
- ❖ If the gain margin GMs equal to one and the phase margin PM is zero degrees, then the control system is **marginally stable**.
- ❖ If the gain margin GM is less than one and / or the phase margin PM is negative, then the control system is **unstable**.

Example:- Draw the nyquist plot and assess the stability of the closed loop system whose open

loop transfer function is $G(s)H(s) = \frac{K}{s(2s+1)}$

$$\begin{aligned} G(j\omega)H(j\omega) &= \frac{K}{j\omega(2j\omega+1)} = \frac{K(1-j2\omega)}{j\omega(1+j2\omega)(1-j2\omega)} \\ &= \frac{K(1-j2\omega)}{j\omega(1+4\omega^2)} = -\frac{2K}{1+4\omega^2} - j\frac{K}{\omega(1+4\omega^2)} \end{aligned}$$

Along the segment (C_1) of the Nyquist contour on the $j\omega$ -axis, s varies from $-j\infty$ to $+j\infty$.

At $\omega = -\infty$,

$$G(j\omega)H(j\omega) = -0 + j0$$

At $\omega = 0^-$,

$$G(j\omega)H(j\omega) = -2K + j\infty$$

At $\omega = 0^+$,

$$G(j\omega)H(j\omega) = -2K - j\infty$$

At $\omega = +\infty$,

$$G(j\omega)H(j\omega) = -0 - j0$$

The infinitesimally small semicircular indent around the pole at the origin of Figure represented by $s = \epsilon e^{j\theta}$ (θ varying from $-\pi/2^\circ$ through 0° to $+\pi/2^\circ$) maps into

$$\lim_{\epsilon \rightarrow 0} [K/(\epsilon e^{j\theta})(2\epsilon e^{j\theta} + 1)] = \lim_{\epsilon \rightarrow 0} K/\epsilon e^{j\theta} = \lim_{\epsilon \rightarrow 0} (K/\epsilon)e^{-j\theta} = \infty e^{-j\theta}$$

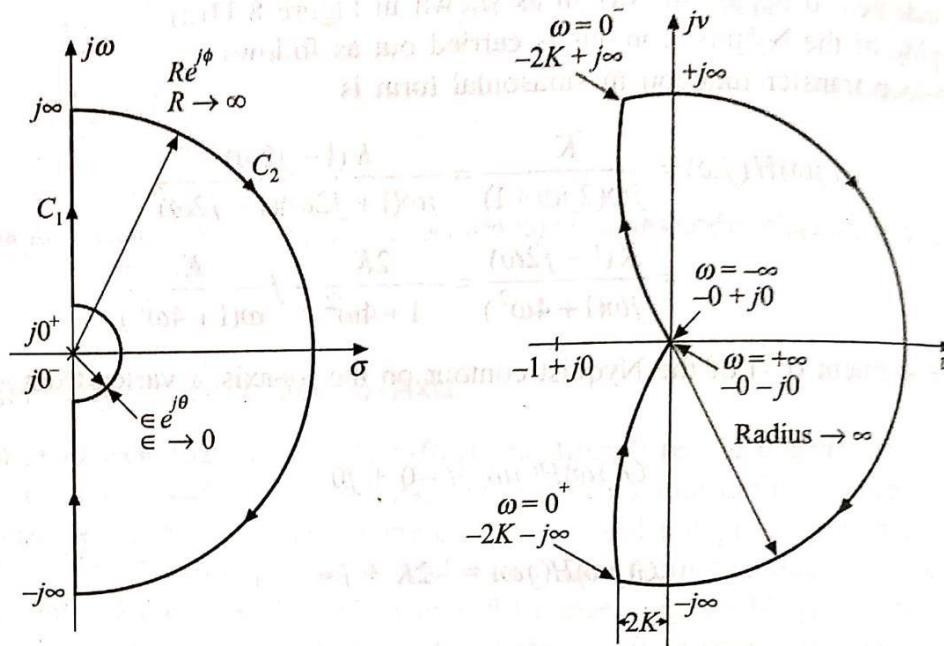
$-\theta$ varies from $+90^\circ \rightarrow 0^\circ \rightarrow -90^\circ$. Thus, the infinitesimal semi-circular indent around the origin of the s -plane maps into a semicircular arc of infinite radius in $G(s)H(s)$ plane extending from $+90^\circ$ through 0° to -90° .

The infinite semicircular arc of the Nyquist contour (segment C_2) of Figure represented by $s = Re^{j\phi}$ (ϕ varying from $+90^\circ$ through 0° to -90°) is mapped into

$$\lim_{R \rightarrow \infty} [K/(Re^{j\phi})(2Re^{j\phi} + 1)] = \lim_{R \rightarrow \infty} K/(2R^2 e^{j2\phi}) = 0e^{-j2\phi}$$

that is, the origin of the $G(s)H(s)$ plane. The $G(s)H(s)$ locus thus turns at the origin with zero radius from -180° through 0° to $+180^\circ$.

The given open-loop system has no poles in the right-half of the s -plane, i.e. $P = 0$. So, per the Nyquist stability criterion, for the closed-loop system to be stable, the Nyquist plot of $G(j\omega)H(j\omega)$ must not encircle the $(-1 + j0)$ point. Since the actual Nyquist plot in Figure 8.11(b) does not encircle the $(-1 + j0)$ point, the closed-loop system is always stable.



EFFECT OF ADDITION OF POLES & ZEROS TO $G(s)H(s)$ ON THE SHAPE OF NYQUIST PLOT

- Addition of poles at $s=0$:** It will affect the stability of the closed loop system adversely. A system that has a loop transfer function with more than one pole at $s=0$ is likely to be unstable or difficult to stabilize.
- Addition of finite non zero pole:** It shifts the phase of nyquist plot by -90° at $\omega = \infty$. The stability is adversely affected.
- Addition of a Zero:** - The effect of addition of zero is to rotate the nyquist plot by 90° in the counter clockwise direction without effecting the value at $\omega = 0$. So it has the effect of reducing the overshoot & the general effect of stabilization.

CONSTANT MAGNITUDE CIRCLE (M- CIRCLE)

$$M(j\omega) = \frac{G(j\omega)}{1+G(j\omega)}$$

$$G(j\omega) = x + jy$$

$$|M(j\omega)| = \frac{\sqrt{x^2 + y^2}}{\sqrt{(1+x)^2 + y^2}}$$

$$M^2(1+x)^2 + M^2y^2 = x^2 + y^2$$

If $M = 1$, then from the above equation we obtain $X = -1/2$. This is the equation of a straight line parallel to the Y-axis and passing through the point $(-1/2, 0)$.

$$x^2(1-M^2) + (1-M^2)y^2 - 2M^2x = M^2$$

Divide both the sides by $(1-M^2)$

$$x^2 + y^2 - 2\frac{M^2}{1-M^2}x = \frac{M^2}{1-M^2}$$

$$\left(\frac{M^2}{1-M^2}\right)^2 \text{ add in both sides}$$

$$x^2 + y^2 - 2x\frac{M^2}{1-M^2} + \left(\frac{M^2}{1-M^2}\right)^2 = \frac{M^2}{1-M^2} + \left(\frac{M^2}{1-M^2}\right)^2$$

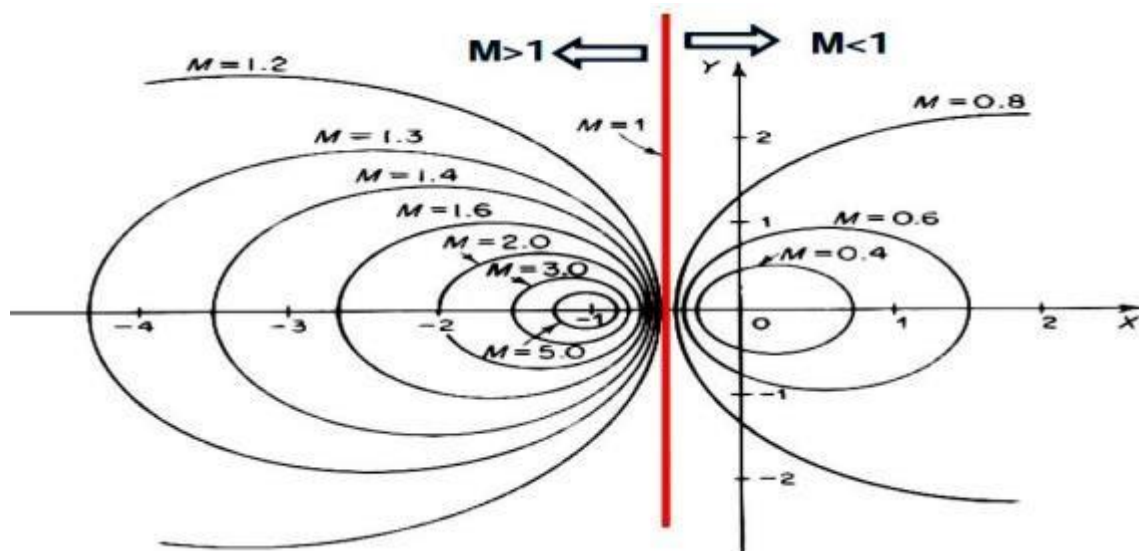
$$\left(x - \frac{M^2}{1-M^2}\right)^2 + (y-0)^2 = \frac{M^2}{1-M^2} + \frac{M^4}{(1-M^2)^2}$$

$$\left(x - \frac{M^2}{1-M^2}\right)^2 + (y-0)^2 = \frac{M^2}{(1-M^2)^2}$$

The above equation represents a family of circles with its center at $\left(\frac{M^2}{1-M^2}, 0\right)$ and radius $\left|\frac{M}{1-M^2}\right|$.

The constant M locii for different value of M. It is clear that:

- The locii are symmetrical wrt to $M=1$
- The M-circles for $M>1$ are on the left side of the line $M=1$ and for $M<1$ the constant M-circles are on the right side of the line $M=1$.



CONSTANT PHASE CIRCLE (N- CIRCLE):

$$\angle M = \alpha = \frac{\angle G(j\omega)}{\angle 1 + G(j\omega)}$$

$$\alpha = \tan^{-1} \frac{y}{x} - \tan^{-1} \frac{y}{1+x}$$

$$N = \tan \left(\tan^{-1} \frac{y}{x} - \tan^{-1} \frac{y}{1+x} \right)$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

Here, $\tan(\alpha) = N$

$$\therefore N = \frac{y}{x^2 + x + y^2} ; \text{ or } N(x^2 + x + y^2) = y$$

$$x^2 + x + y^2 - \frac{y}{N} = 0$$

Add $\frac{1}{4} + \frac{1}{(2N)^2}$ to both sides, we get

$$x^2 + x + y^2 - \frac{y}{N} + \frac{1}{4} + \frac{1}{(2N)^2} = \frac{1}{4} + \frac{1}{(2N)^2}$$

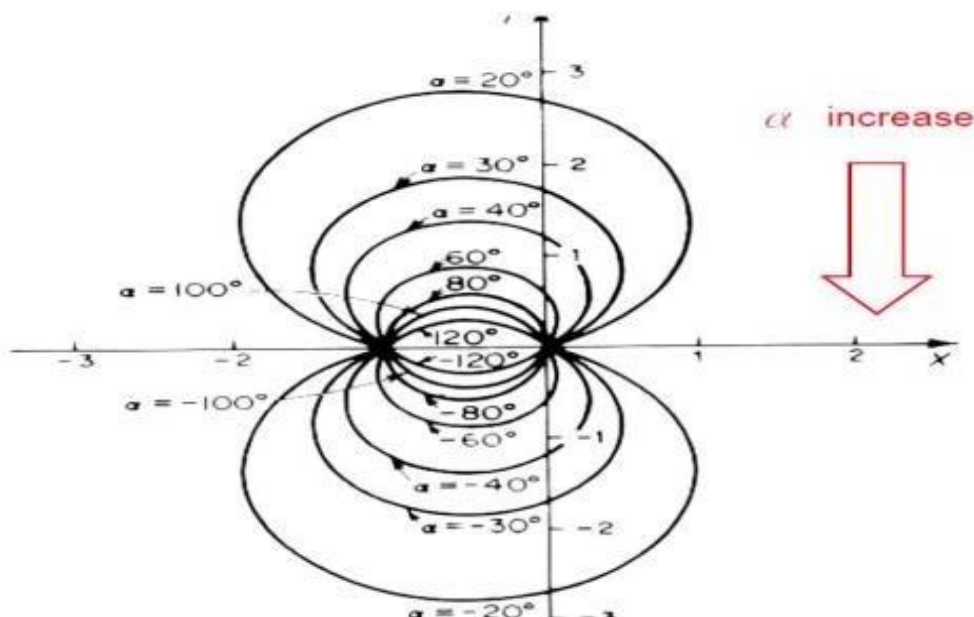
$$\text{or } (x + 1/2)^2 + \left(y - \frac{1}{2N}\right)^2 = \frac{1}{4} + \frac{1}{(2N)^2}$$

The above equation represents a family of circles with its center at $(-\frac{1}{2}, \frac{1}{2N})$ and radius

$$\sqrt{\frac{1}{4} + \left(\frac{1}{2N}\right)^2}$$

It is observed that

- The centre is lying always at a distance $x = -1/2$ and y depends upon the phase shift.
- All the circles passes through -1 as well as 0.



NICHOLS CHART

- The chart consisting of constant-magnitude loci and constant phase-angle loci in the log-magnitude versus phase diagram is called Nichols chart.
- The critical point $(-1+j0)$ is mapped to the Nichols chart as the point $(0 \text{ dB}, 180\text{degree})$. The Nichols chart contains curves of constant closed-loop magnitude and phase angle.
- The designer can graphically determine the phase margin, gain margin, resonant peak magnitude, resonant peak frequency, and bandwidth of the closed loops system from the plot of the open-loop locus.
- The Nichols chart is symmetric about -180 degree axis. The constant-magnitude loci and constant phase-angle loci repeat for every 360 degree, and there is a symmetry at every 180 degree. The constant-magnitude loci are centred about the critical point $(0 \text{ dB}, -180 \text{ degree})$.
- The intersection of the open-loop frequency response curve and the constant-magnitude loci and constant phase-angle loci give the values of the magnitude and the phase angle of the closed loop frequency response at each frequency point.
- If the open-loop frequency response curve does not intersect the constant-magnitude loci but is tangent to it, then the resonant peak value of the closed-loop frequency response is given by that loci. The resonant peak frequency is given by the frequency at the point of tangency.
- The phase crossover point is the point where the open-loop locus intersects the -180 degree axis, and the gain crossover point is the point where the locus intersects the 0 dB axis.
- The phase margin is the distance (measured in degrees) between the gain crossover point and the critical point $(0 \text{ dB}, -180 \text{ degrees})$.
- The gain margin is the distance (in decibels) between the phase crossover point and the critical point. The frequency at the intersection of the open-loop locus and the -3 dB locus gives the bandwidth.

